A STUDY OF THE CHARACTERISTICS OF CONTROL SYSTEMS DESIGNED USING THE QUADRATIC INDEX OF PERFORMANCE

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Approved:

Co-Chairman

Co-Chairman

Date approved by Chairman: ______

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SUMMARY

The primary objective of this investigation is to study the weighted quadratic (in the state and control variables) index of performance (IP) in an attempt to determine its applicability as a design criterion for automatic control systems.

The quadratic IP may be written in vector notation as:

$$IP = \frac{1}{2}\int_{o}^{T} [\langle\overline{x},C\overline{x}\rangle+\langle\overline{u},G\overline{u}\rangle]\,dt$$

where $\overline{x}$ is the state vector, $\overline{u}$ is the control vector, C is the weighting coefficient matrix for the state variables, and G is the weighting coefficient matrix for the control variables. In this investigation, the C matrix is restricted to being a diagonal matrix, and the G matrix is restricted to being the identity matrix. As a result of these restrictions on C and G, only squares of the state and control variables appear in the integral of IP. There are no cross products of the state or control variables in the integrand.

The class of quadratic IP described above is applied to three classes of problems as follows:

Class I. Constant coefficient linear plants optimized over an infinite time interval.

Class II. Constant coefficient linear plants

optimized over a finite time interval.

Class III. Non-linear plants optimized over a finite time interval.

For problems of Class I, the relationships between the plant parameters, the elements of matrix C and the roots of the characteristic equation of the "optimum" system are developed analytically. It is shown that the roots of the characteristic equation of the optimum system are those roots with negative real parts of

$$\lambda^{2n} - M_n\lambda^{2(n-1)} + M_{n-1}\lambda^{2(n-2)} + \cdots + (-1)^{k+1}M_{n-k}\lambda^{2(n-k-1)} + \cdots + (-1)^n M_1 = 0$$

$$(k = 0, 1, \cdots, n-1)$$

where the M_{n-k} are polynomials in the coefficients of the plant characteristic equation and the elements of the C matrix. The M_{n-k} are given for plants of order 2, 3, and 4. Methods of determining the M_{n-k} for plants of any order are presented.

The effect of the M_{n-k} upon the roots of the characteristic equation of the optimum system is studied in general, and numerical results are presented for second and third order plants.

Since definite relationships exist between the roots of the optimum system characteristic equation, the plant parameters, and the weighting coefficients, it is possible to specify weighting coefficients on the basis of the root locations desired for the optimum system. However, if root locations are specified, there is no need to carry out an optimization procedure, because the feedback gains are fixed by root locations and plant parameters. Therefore, it is concluded that the quadratic IP is of little value in design problems of Class I. This statement holds only so long as no general "a priori" methods of specifying the weighting coefficients exist, which appears to be the case at present.

Problems of Classes II and III are investigated by using the analytical results for Class I problems as a basis for selecting the weighting coefficients of the quadratic IP for Class II and III problems.

It is shown that for problems of Class II, the controller is linear in the state variables with time varying coefficients. A general result for those cases in which the plant is of second order is presented. It is concluded that when the weighting coefficients are large, the optimum system response is essentially the same as could be obtained from a constant gain controller. Only for small values of the weighting coefficients does the optimum system exhibit response characteristics which are clearly superior to the characteristics of constant gain controllers.

An example problem of Class III is solved by linearizing the plant differential equation in some "reasonable" manner, and then selecting weighting coefficients for this linearized model. The resulting system does have satisfactory response characteristics. It is concluded that the quadratic IP can result in an acceptable design for problems of Class III.

CHAPTER I

INTRODUCTION

Objective

The primary objective of this investigation is to study the weighted quadratic (in the state and control variables) index of performance (IP) in an attempt to determine its applicability as a design criterion for automatic control systems.

The quadratic IP in the state and control variables may be written in vector notation as

$$IP = \frac{1}{2} \int_{o}^{T} [\langle \bar{x}, C\bar{x} \rangle + \langle \bar{u}, G\bar{u} \rangle] \, dt \qquad (1)$$

where x and u are the vector representations of the state and control variables, C is a matrix of weighting coefficients for the state variables and G is a matrix of weighting coefficients for the control variables. The symbol $\langle \bar{a}, \bar{b} \rangle$ denotes the scalar product of the vectors $\bar{a}$ and $\bar{b}$.

Background

The term "index of performance" has not yet been given any widely accepted definition in the literature. Many authors define the term for the purposes of a particular study, while others simply use the term generically, assuming that the reader can attach significance to the term without any precise definition. The terms "performance measure" and "performance criterion" are also used in the literature to imply essentially the same ideas as "index of performance." Since there is no generally accepted explicit definition of the term, "index of performance" will be defined here as any quantity whose value is related to the dynamic performance of a physical system. The term "dynamic performance" includes within its scope transient characteristics, frequency response characteristics and energy consumption characteristics of the system.

The history of indices of performance for automatic control systems properly begins with a discussion of systems which can be described by linear constant-coefficient differential equations. The most basic performance measure applicable to linear constant-coefficient control systems is the measure of the stability of the system.[1] A control system is said to be stable if the system, operating at some steady-state condition, will return to that condition after

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[1]The concept of stability can, of course, be applied to any type of control system. Furthermore, several types of stability have been defined in the literature. Kalman and Bertram discuss stability in some detail in (1).

a disturbance. The characteristic equation of a linear constant-coefficient system may be examined to determine whether or not the system is stable. Routh (2) and Nyquist (3) have developed methods of determining stability of linear systems. Lyapunov (4) has given a method of determining stability which is applicable to systems described by linear equations with time varying coefficients and by non-linear equations, as well as to linear constant-coefficient systems.

A wide variety of indices of performance applicable to linear constant-coefficient systems has been in use for many years. Gibson (5) gives an excellent summary of these IP along with comments upon their relative merits. These classical IP may be divided into a time domain group and a frequency domain group.

The time domain group of IP includes maximum overshoot, rise time and settling time for a step input to the system.[2] For systems of third and higher order, the time domain IP are usually evaluated with the aid of either an analog or a digital computer. For systems of first and

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[2]These measures are also applicable to time-varying linear and to non-linear systems, of course. However, they are basic to linear constant-coefficient system analysis and design.

second order the characteristic equation of the system may be examined to determine the time domain IP. For example, the characteristic equation of linear second order constant coefficient systems may be written in the form:

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0 \tag{2}$$

The values of ζ, ω_n and $\zeta\omega_n$ determine what the maximum overshoot, rise time, and settling time will be for the system. The roots of the characteristic equation may also be used to estimate the value of the time domain performance index for systems of any order. In fact, examination of the roots of the characteristic equation is one of the more useful methods of estimating the time domain IP, and any experienced control engineer can determine the desirability of a linear system from knowledge of these roots.

Frequency domain IP are based upon the steady-state performance of a system subjected to a sinusoidally varying input. The frequency domain performance may be related to the time domain performance through Fourier transforms, and therefore, the frequency domain characteristics of a system imply certain time domain characteristics. The usefulness of frequency domain IP is due primarily to the fact that convenient methods of determining these characteristics have been developed. Nyquist (3) was the first to suggest useful frequency domain IP and to give convenient methods for evaluating these indexes. Phase margin and gain margin were suggested by Nyquist (3) as frequency domain IP. Bode (6) in 1945 suggested an improved method of determining phase margin and gain margin. Bode's method also yields information as to the resonance peak and the bandwidth, which are also useful frequency domain IP.

The frequency domain and time domain indices of performance have been invaluable in the design of automatic control systems in the past. However, these indices of performance virtually always require that the "cut and try" method of design be used for systems of higher than second order. That is, a given system is modified and the index of performance evaluated. If the index of performance is not satisfactory, the system is again modified and the index evaluated. This "cut and try" process is repeated until an acceptable system results.

In recent years there has appeared in the literature a large amount of work devoted to specifying indices of performance that have the characteristics of a metric quantity. That is, efforts have been made to specify indices of performance for which the desirability of the system increases as the value of the IP increases (or decreases). IP of this metric type offer the possibility of the synthesis of "optimum" systems by mathematically maximizing (or minimizing) the IP. If suitable IP of this type could be found, the "cut and try" methods could be replaced by a single extremizing operation which would yield not just an acceptable system but an "optimum" system. Many individuals in the control field have recognized the desirability of IP that lend themselves to mathematical optimization, and a rather large number of IP have been suggested in recent years. Most of these relatively new IP are in the form of an integral of some time varying function over an interval of time. Hall (7) appears to have been the first to suggest an integral IP. He proposed the following index as a measure of system performance:

$$IP = \int_0^\infty e^2(t)\ dt \tag{3}$$

where e(t) is the difference between the desired output and the actual output of the system. The optimum system would be that system yielding the minimum value of IP. Since Hall's original use of an integral IP, numerous other integral indices have been proposed (8-14). An excellent summary of the indices proposed through 1960 is given by Schultz and Rideout (13). Wolkovitch, et al., (14) also give a tabular summary of the integral IP that have been proposed. Table 1, page 7, gives some of the integral IP

Table 1

Some Proposed Integral IP

Symbol	Measure	Comments
ITAE	$\int_0^T t \, \lvert e(t) \rvert \, dt$	According to ref (14), this is good measure for analog studies.
ITSE	$\int_0^T t \, e^2(t) \, dt$	Better for analytical work than ITAE according to (14).
LSE	$\int_0^T e^2(t) \, dt$	Yields result only for form determined systems.
LAE	$\int_0^T \lvert e(t) \rvert \, dt$	Yields acceptable results only for low order systems.
IQSC	$\int_0^T [\langle \bar{x}, C\bar{x} \rangle + \langle \bar{u}, G\bar{u} \rangle] dt$	Mathematically convenient, but little information as to how to select C and matrices.
IFET	$\int_0^T F \, [e(t), t] dt$	Quite "general" in that it includes all other measures above.

which have been proposed and indicates letter symbols that will be used to refer to these indices.

The selection of an appropriate IP is obviously an essential step in the design (or synthesis)of an optimum control system. Nevertheless, there has been very little published information dealing with this problem. Wolkovitch, et al.(14), have investigated the integral of time-absolute error (ITAE) and the integral of time-squared error (ITSE) indices in some detail, and their work is a significant contribution to the literature on selection of IP. Kalman (15) has investigated the quadratic in the state and control variables (IQSC) in order to determine when a linear system is optimal. Kalman shows that a stable linear system is optimal for some IQSC. He also investigated the optimum system for those cases in which the weightings of the control variables approach infinity. Merriam (16) uses the IQSC extensively in his book and gives some heuristic arguments concerning selection of the weighting coefficients.

The quadratic IP has appeared rather frequently in the literature as the basis for the optimization of specific plants. However, the solutions which have been presented assume that the weighting coefficients have been given. There does not appear to be a single case in which concrete reasons are presented for selecting the particular weighting coefficients. As a matter of fact, it appears that the primary reason for selecting the quadratic IP in optimization problems has been the mathematical tractability of that IP. While mathematical tractability is certainly a desirable (even necessary) characteristic of an IP, it is not of itself sufficient justification for choosing a particular index. The selected index should also give some indication of the performance that can be expected of the resulting system. In other words, selection of an IP should be based in part upon what operating characteristics will be possessed by the resulting system.

Although the literature of control theory contains many examples of design using a quadratic IP, there is virtually no information on selection of the weighting coefficients for a particular problem. Since the quadratic IP does have desirable mathematical properties (as evidenced by its frequent appearance), it seems that it should be investigated further in order to determine methods of specifying the weighting coefficients. If such methods can be found, it is possible that engineering applications of the theory could then be made.

Scope of the Investigation

The analytical investigation of the quadratic IP will be confined to linear constant coefficient plants with a single input variable, optimized over an infinite time interval. An attempt will be made to determine relations between the roots of the optimum system characteristic equation, and the weighting coefficients and plant parameters. As was mentioned previously, an experienced control engineer can determine the desirability of a control system given the location of the roots of its characteristic equation. Therefore, if the effect of the weighting coefficients upon these roots is known, then a basis for selecting the weighting coefficients is available, at least in principle.

Once the effect of weighting coefficients upon linear constant-coefficient plants optimized over an infinite time interval is known, it is possible that this information would aid in selecting weighting coefficients for the optimization of linear constant coefficient plants optimized over a finite time interval, linear time-varying plants, and non-linear plants. Specific examples are included which tend to verify this supposition.

CHAPTER II

ANALYTICAL INVESTIGATION

It was noted in Chapter I that there exists very little information to guide one in selecting an integral type IP. The IP selection problem involves first choosing one of the several forms of IP that have been suggested and then specifying any arbitrary parameters that appear in the IP. For example, if the quadratic IP is selected as a design tool because of its mathematical convenience, the designer must specify values for the weighting coefficients in order to carry out the design computations.

This analysis will be devoted to determining the effects of the weighting coefficients of the quadratic IP upon the characteristics of the final system for a class of linear plants.

Formulation of the Problem

The design of control systems for plants governed by constant-coefficient linear differential equations will be considered. The class of plants will be further restricted to those having a single input variable. The integral of a weighted quadratic in the state and control variables extending over an infinite time range will be the IP upon which the design will be based. The only restriction on the state variables will be that of boundedness.

Stated in terms of mathematical relations the problem is as follows.

Given a plant governed by a differential equation of the form

$$\frac{d^n x}{dt^n} + b_n \frac{d^{n-1}x}{dt^{n-1}} + \cdots + b_2 \frac{dx}{dt} + b_1 x = u \qquad (4)$$

where the $b_i (i=1,2,\cdots,n)$ are constants and u is the input, and a quadratic IP to be minimized of the form

$$IP = \frac{1}{2}\int_0^T \left\{ c_1 x^2 + c_2 \left(\frac{dx}{dt}\right)^2 + \cdots + c_n \left(\frac{d^{n-1}x}{dt^{n-1}}\right)^2 + u^2 \right\} dt \qquad (5)$$

determine the effect of the $c_i (i=1,2,\cdots,n)$ upon the resulting system for the case in which $T \to \infty$. It is further required that the plant output, x(t) and all of its derivatives be bounded. The control variable u is allowed to take on any finite values.

Solution

The maximum principle of L. S. Pontryagin (17) will be used to solve the problem. It is convenient to convert to state vector notation before proceeding with the solution. This conversion is accomplished in the following manner.

Define the n quantities

$$x_1 = x \tag{6}$$

$$x_2 = \frac{dx}{dt}$$

$$\vdots$$

$$x_n = \frac{d^{n-1}x}{dt^{n-1}}$$

These definitions allow the formulation of the system of first order differential equations

$$\dot{x}_1 = x_2 \tag{7}$$

$$\dot{x}_2 = x_3$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = -b_1x_1 - b_2x_2 - \cdots - b_nx_n + u$$

The state vector $\overline{x}(t)$ is defined by

$$\overline{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{8}$$

A control vector $\overline{u}(t)$ is defined by

$$\overline{u}(t) = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \tag{9}$$

where u_i $(i=1,2,\cdots,n)$ is the input component appearing in the i^{th} row of the vector form of the system of equations governing the plant. In this case, $u_n = u$ and the u_i $(i=1,2,\cdots,n-1)$ are zero. By defining the n x n matrices

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -b_1 & -b_2 & \cdots & \cdots & \cdots & -b_n \end{bmatrix} \tag{10}$$

and

$$D = \begin{bmatrix} 0 & & & & & & & \\ & 0 & & & & & \bigcirc & \\ & & 0 & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & \bigcirc & & & & & 0 & \\ & & & & & & & 1 \end{bmatrix} \tag{11}$$

the differential equation (4) may be converted to the vector differential equation

$$\dot{\overline{x}} = B\overline{x} + D\,\overline{u} \tag{12}$$

Equation (5) may also be written in vector notation by defining the n x n constant matrices

$$C = \begin{bmatrix} c_1 & & & & & \\ & c_2 & & & \bigcirc & \\ & & \cdot & & & \\ & & & \cdot & & \\ & \bigcirc & & & \cdot & \\ & & & & & c_n \end{bmatrix} \tag{13}$$

and

$$G = \begin{bmatrix} 1 & & & & \\ & 1 & & \bigcirc & \\ & & \cdot & & \\ & & & \cdot & \\ & \bigcirc & & & \cdot \\ & & & & 1 \end{bmatrix} = I \tag{14}$$

The performance measure becomes

$$IP = \frac{1}{2}\int_{o}^{T} [<\bar{x},C\bar{x}> + <\bar{u},G\bar{u}>]dt \tag{15}$$

where <,> denotes the scalar product of two vectors.

Application of Pontryagin's maximum principle requires definition of a costate vector ($\bar{p}$) and the Hamiltonian (H). The costate vector is an n - dimensional vector.

$$\bar{p} = \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{bmatrix} \tag{16}$$

The Hamiltonian is by definition

$$H = <\bar{p},\dot{\bar{x}}> - \frac{1}{2}[<\bar{x},C\bar{x}>+<\bar{u},G\bar{u}>] \tag{17}$$

and the costate vector is defined to be the vector satisfying the relations

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} \quad (i = 1, 2, \cdots, n) \tag{18}$$

This definition of the costate vector may be written in vector notation as

$$\dot{\overline{p}} = -\nabla_{\overline{x}} H \tag{19}$$

Pontryagin's maximum principle states that in order for IP to take on its minimum value, it is necessary that $\overline{u}$ be chosen so that H will take on its maximum value at every t in the interval $0 \leq t \leq T$. In this case (since $\overline{u}$ is not bounded), this requirement becomes

$$\nabla_{\overline{u}} H = 0 \tag{20}$$

Collina and Dorato (18) have shown that application of the maximum principle to this type problem results in the equation for $\overline{u}$

$$\overline{u} = G^{-1} D^T \overline{p} \tag{21}$$

The system of vector differential equations that must be solved becomes

$$\begin{aligned} \dot{\overline{x}} &= B\overline{x} + G^{-1} D^T \overline{p} \\ \dot{\overline{p}} &= C\overline{x} - B^T \overline{p} \end{aligned} \tag{22}$$

subject to the boundary condition

$$\lim_{T\to\infty} x_i(T) < \infty \; (i = 1, 2, \cdots, n) \tag{23}$$

Equations (22) can be rewritten as

$$\begin{aligned} \dot{\overline{x}} &= B\overline{x} + D\,\overline{p} \\ \dot{\overline{p}} &= C\overline{x} - B^T\overline{p} \end{aligned} \tag{24}$$

by making use of the relations

$$D = D^T \tag{25}$$

and

$$G^{-1} = G = I \tag{26}$$

which can be seen to be true from equations (11) and (14).

The solution of equations (24) subject to conditions (23) yields the optimum trajectory for $\overline{x}(t)$. The equation of the optimum controller $\overline{u}(t)$ can also be found from the solution of (24) by substituting $\overline{p}(t)$ into equation (21).

The solution of equation (24) subject to the condition given by (23) and the determination of the optimum controller follow below.

The 2n x 2n partitioned matrix A is defined by

$$A = \begin{bmatrix} B & D \\ C & -B^T \end{bmatrix} \tag{27}$$

The vector equations (24) may now be written as the single vector equation

$$\begin{bmatrix} \dot{\overline{x}} \\ \dot{\overline{p}} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{p} \end{bmatrix} \tag{28}$$

The general solution of the vector equation (28) depends solely upon the properties of the matrix A. For the class of problems being considered here, the submatrices C and D of A are symmetric. It can be shown (see Appendix A) that if C and D are symmetric, then the matrix A has eigenvalues which occur only in oppositely signed pairs.

Furthermore, necessary conditions that the eigenvalues of A all have non-zero real parts may be determined by examining the characteristic equation of A. For the present, it will be assumed that all eigenvalues of A have non-zero real parts.

Let λ_1, λ_2, $\cdots$, λ_n, be the eigenvalues of A with positive real parts. Here any repeated eigenvalue is entered into the list k times, where k is the multiplicity of the repeated eigenvalue. Because of the oppositely signed pairs feature of the eigenvalues, the matrix A will have also the eigenvalues $-\lambda_1, -\lambda_2, \cdots, -\lambda_n$. The solution to the system of equations (28) can be written in the form (see reference (19)).

$$x_j(t) = \sum_{i=1}^{r} \left\{ \left[\sum_{k=o}^{m_i-1} a_{j,i}^{(k)} \; t^k \right] e^{\lambda_i t} \right\} + \tag{29}$$

$$\sum_{i=1}^{r} \left\{ \left[\sum_{k=o}^{m_i-1} a_{j,-i}^{(k)} t^k \right] e^{-\lambda_i t} \right\} \quad (j=1,2,\cdots,n)$$

$$p_j(t) = \sum_{i=1}^{r} \left\{ \left[\sum_{\ell=o}^{m_i-1} a_{j,i}^{(\ell)} t^{\ell} \right] e^{\lambda_i t} \right\} + \tag{30}$$

$$\sum_{i=1}^{r} \left\{ \left[\sum_{\ell=o}^{m_i-1} a_{j,-i}^{(\ell)} t^{\ell} \right] e^{-\lambda_i t} \right\} \quad (j=1,2,\cdots, n)$$

where 2r is the number of distinct eigenvalues of A, m_i the multiplicity of the i^{th} root, and the a's are constants (not all independent) which may be evaluated from the boundary conditions.

Since it is required that

$$\lim_{T\to\infty} \left| x_j(T) \right| < \infty \tag{23}$$

$$j = (1,\ 2,\ \cdots,\ n)$$

it is immediately apparent that the boundary condition at time infinity requires that the coefficients of the $e^{\lambda_i t}$ terms in equation (29) be zero. That is, it is necessary that

$$a^{(k)}_{j,i} = 0 \tag{31}$$

must hold in equation (29). The state variables are then given by

$$x_j(t) = \sum_{i=1}^{r} \left\{ \left[\sum_{k=o}^{m_i-1} a^{(k)}_{j,-i} t^k \right] e^{-\lambda_i t} \right\} \tag{32}$$

It can be noted at this point that the desired result in the design of an automatic control system is simply a differential equation, or perhaps a system of differential equations. From the first of equations (32) it can be seen that $x_1(t)$ is the solution of a differential equation whose characteristic equation is given by

$$(s+\lambda_1)\ (s+\lambda_2)\ \cdots\cdot\ (s+\lambda_n) = 0 \tag{33}$$

where roots of multiplicity m_i appear m_i times in the product (33). A differential equation in $x_1(t)$ whose characteristic equation is given by (33) is

$$\frac{d^n x_1}{dt^n} + \left[\sum_{i=1}^{n} \lambda_i\right] \frac{d^{n-1} x_1}{dt^{n-1}} + \left[\sum_{i=1}^{n-1} \lambda_i \sum_{j=i+1}^{n} \lambda_j\right] \frac{d^{n-2} x_1}{dt^{n-2}} \tag{34}$$

$$+ \cdots + \left[\sum_{i=1}^{n-k} \lambda_i \sum_{j=i+1}^{n-k+1} \lambda_j \sum_{\ell=j+1}^{n-1+2} \lambda_\ell \cdots\right] \frac{d^{n-k-1} x_1}{dt^{n-k-1}}$$

$$+ \cdots \left[\prod_{i=1}^{n} \lambda_i\right] x_i = 0 \quad (k=0,\ 1,\ \cdots,\ n-1)$$

where in the general term there are k+1 summations.

Recalling that x_1 equals x and comparing equation (34) with equation (4)

$$\frac{d^n x}{dt^n} + b_n \frac{d^{n-1} x}{dt^{n-1}} + \cdots + b_2 \frac{dx}{dt} + b_1 x = u \tag{4}$$

it can be seen that u must be given by

$$u(t) = \left[b_1 - \prod_{i=1}^{n} \lambda_i\right] x(t) + \left[b_2 - \sum_{i=1}^{n-1} \lambda_i \sum_{j=i+1}^{n} \lambda_j\right] \frac{dx(t)}{dt} \tag{35}$$

$$+ \cdots + \left[b_{n-k} - \sum_{i=1}^{n-k} \lambda_i \sum_{j=i+1}^{n-k+1} \lambda_i \cdots\right] \frac{d^{n-k-1} x(t)}{dt^{n-k-1}}$$

$$+ \cdots + \left[b_n - \sum_{i=1}^{n} \lambda_i\right] \frac{d^{n-1} x(t)}{dt^{n-1}} \quad (k=0,\ 1,\ \cdots,\ n-1)$$

where there are k + 1 summations in the general term.

Equation (35) gives the control u(t) required to minimize the selected IP (equation (5)). The control given by (35) is said to be the "optimum" control for the IP (5) when the state variables are required to remain finite. It can be seen from equation (35) that the control function u(t) is a linear combination of the n state variables (defined in equation (6)), and that the coefficients of the state variables in (t) are constant. Chang Jen-Wei (20) has shown previously that the control is a linear function of the state variables with constant coefficients, but he gives no explicit relationships. Equation (35) gives explicit relationships between the system parameters, the eigenvalues of the matrix A, and the coefficients of the state variables in the control equation.

The configuration of the optimum system is shown in Figure 1. The constants K_{n-k} in Figure 1 are defined by

$$K_{n-k} = b_{n-k} - \sum_{i=1}^{n-k} \lambda_i \sum_{j=i+1}^{n-k+1} \lambda_i \cdots\cdots \tag{36}$$

$$(k=0,\ 1,\ \cdots,\ n-1)$$

where there will be k+1 summations in the general term.

Referring again to equation (34), it can be seen that the roots of the characteristic equation of the optimum

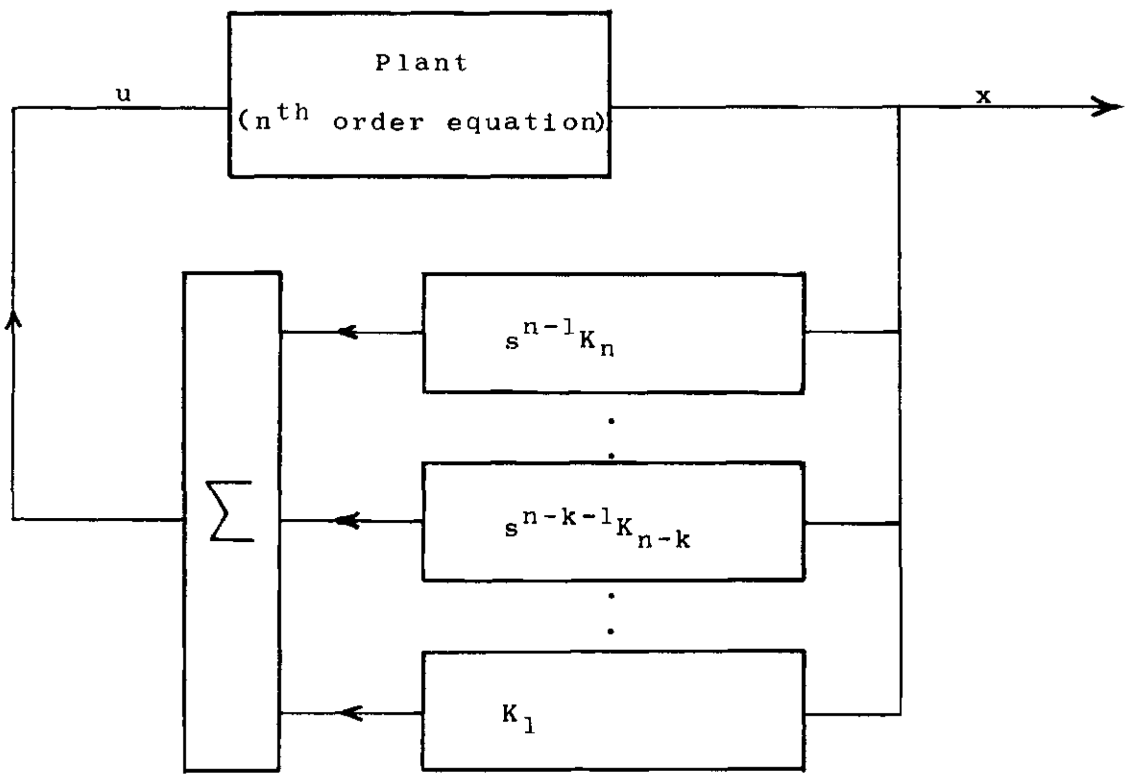

Figure 1. Optimum System Configuration.

system are determined by the eigenvalues of the matrix A, and the roots are those eigenvalues of A that have negative real parts.

It is now possible to determine the relations between the weighting constants c_i, the plant parameters b_i and the roots of the characteristic equation of the optimum system. Since the matrix A has eigenvalues which occur only in oppositely signed pairs, the characteristic equation of A is an even polynomial function in λ. Only even powers of λ can appear in an even polynomial function. Therefore, the characteristic equation of A may be written in the form

$$\lambda^{2n} - M_n \lambda^{2(n-1)} + \cdots + M_{n-k}(-1)^{k+1} \lambda^{2(n-k-1)} + \cdots + M_1(-1)^n = 0 \qquad (37)$$

$$(k=0,\ 1,\ \cdots,\ n-1)$$

where the M_{n-k} are polynomials in the b_i and the c_i. The M_{n-k} may also be expressed as functions of the c_i and the roots of the plant characteristic equation or as functions of the eigenvalues of the A matrix. While it is possible to deal with the general case for any value of n, it is more convenient at this point to consider particular values of n and to infer general results from these particular cases. The M_{n-k} have been determined in the three forms given above for the cases $n = 2, 3, 4$.

The M_{n-k} as functions of the b_i and c_i were found by expanding the determinant $|A - \lambda I|$ for the values of n given above.

In order to obtain the M_{n-k} as functions of the roots of the characteristic equation of the plant, equation (4) was written in the operational form

$$[(s+a_1)\ (s+a_2)\ \cdots\ (s+a_n)]\, x(t) = u(t) \qquad (38)$$

where $s = \frac{d}{dt}$. The coefficient of $x(t)$ in equation (38) was then expanded, and the resulting polynomial in s was compared with equation (4) to obtain relationships between the b_i and the a_j $(i, j = 1, 2, \cdots, n)$. These relationships were then used to eliminate the b_i from the equations for the M_{n-k} in terms of the b_i and c_i.

The functional relationships between the M_{n-k} and the eigenvalues of the A matrix were found by writing equation (37) in the factored form

$$(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_2^2)\ \cdots\ (\lambda^2-\lambda_n^2) = 0 \qquad (39)$$

and expanding equation (39). The resulting polynomial was then compared term by term with equation (37) to obtain the M_{n-k} as functions of the λ_i.

Table 2, page 27, gives the M_{n-k} as functions of the

Table 2

Equivalent Expressions for the M_i $(i = 1, 2, \cdots, n)$

For Systems of Second, Third, and Fourth Order

n	M_1	M_2	M_3	M_4
2	$b_1^2 + c_1$	$b_2^2 + c_2 - 2b_1$		
	$a_1^2\, a_2^2 + c_1$	$a_1^2 + a_2^2 + c_2$		
	$\lambda_1^2\, \lambda_2^2$	$\lambda_1^2 + \lambda_2^2$		
3	$b_1^2 + c_1$	$b_2^2 + c_2 - 2b_1 b_3$	$b_3^2 + c_3 - 2b_2$	
	$a_1^2\, a_2^2\, a_3^2 + c_1$	$a_1^2\, a_2^2 + a_1^2\, a_3^2 + a_2^2\, a_3^2 + c_2$	$a_1^2 + a_2^2 + a_3^2 + c_3$	
	$\lambda_1^2\, \lambda_2^2\, \lambda_3^2$	$\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2$	$\lambda_1^2 + \lambda_2^2 + \lambda_3^2$	

Table 2 (continued)

n	M_1	M_2	M_3	M_4
4	$b_1^2 + c_1$	$b_2^2 + c_2 - 2b_1b_3$	$b_3^2 + c_3 + 2b_1 -$ $2b_2b_4$	$b_4^2 + c_4 - 2b_3$
	$a_1^2a_2^2a_3^2a_4^2 + c_1$	$a_1^2a_2^2a_3^2 + a_1^2a_2^2a_4^2 +$ $a_1^2a_3^2a_4^2 + a_2^2a_3^2a_4^2 +$ c_2	$a_1^2a_2^2 + a_1^2a_3^2 +$ $a_1^2a_4^2 + a_2^2a_3^2 +$ $a_2^2a_4^2 + a_3^2a_4^2 + c_3$	$a_1^2 + a_2^2 + a_3^2 +$ $a_4^2 + c_4$
	$\lambda_1^2\ \lambda_2^2\ \lambda_3^2\ \lambda_4^2$	$\lambda_1^2\lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_2^2\lambda_4^2 +$ $\lambda_1^2\lambda_3^2\lambda_4^2 + \lambda_2^2\lambda_3^2\lambda_4^2$	$\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 +$ $\lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 +$ $\lambda_2^2\lambda_4^2 \quad \lambda_3^2\lambda_4^2$	$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 +$ λ_4^2

three groups of parameters mentioned above for $n = 2, 3, 4$.

Relationships between the λ_i, the c_i, and the plant parameters may be found by equating equivalent expressions for the M_{n-k}. Setting the M_{n-k} as functions of the a_i and c_i equal to the M_{n-k} as functions of the λ_i allows the inference of the following general relationships between the λ_i, the c_i and the a_i.

$$c_n + \sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} \lambda_i^2 \tag{40}$$

$$c_{n-1} + \sum_{i=1}^{n-1} a_i^2 \sum_{j=i+1}^{n} a_j^2 = \sum_{i=1}^{n-1} \lambda_i^2 \sum_{j=i+1}^{n} \lambda_j^2$$

$$\vdots$$

$$c_{n-k} + \sum_{i=1}^{n-k} a_i^2 \sum_{j=i+1}^{n-k+1} a_j^2 \cdots = \sum_{i=1}^{n-k} \lambda_i^2 \sum_{j=i+1}^{n-k-1} \lambda_j^2 \cdots\cdots$$

$$\vdots$$

$$c_1 + \prod_{i=1}^{n} a_i^2 = \prod_{i=1}^{n} \lambda_i^2$$

Equations (40) show implicitly the effect of the c_i upon the roots, $-\lambda_i$, of the characteristic equation of the optimum system in terms of the roots, $-a_i$, of the plant characteristic equation. The characteristic equation of the A matrix (equation (37)) gives an equivalent result in terms of the coefficients b_i of the plant differential equation. From Table 2, it can be observed that

$$M_{n-k} = c_{n-k} + f_k (b_1, b_2, \cdots, b_n) \tag{41}$$

$$(k = 0, 1, \cdots, n-1)$$

where $f_k (b_i)$ denotes a polynomial function. It is obvious that the question of the effect of the c_i upon the roots of the optimum system is equivalent to the problem of determining the effect the individual coefficients of a polynomial upon the roots of a polynomial. It is also obvious that any desired characteristic equation of the optimum systems may be obtained by appropriate choices of the c_i.

Results

The results obtained thus far may be summarized in the following statement:

For the class of problems where the plant can be described by a constant coefficient linear differential equation, and a quadratic IP of the form of equation (5) is used as a basis of design, the roots of the characteristic equation of the optimum system are those roots of equation (37) which have negative real parts.

This result may be considered from two points of view. First, given the plant parameters and the weighting coefficients for a design problem of the class being considered, the optimum controller may be determined simply by solving for the roots of equation (37) and evaluating the coefficients of equation (35). On the other hand, the result can be considered from the standpoint of what effects the choice of the c_i have upon the optimum system. This second viewpoint is the one of most interest here. As mentioned previously, the question of the effects of the c_i upon the roots of the optimum system is in reality the problem of the effect of the coefficients of a polynomial upon the roots of that polynomial. This problem has received a good deal of attention for some years, and a large number of theorems regarding bounds on the roots of a polynomial as a function of the coefficients has appeared in the literature. See for example reference (21). However, very few of these theorems are of value in determining the effects of the c_i upon the roots of the optimum system. Nevertheless, some general observations are possible.

Since the roots of the characteristic equation of the optimum system can be determined from equation (37), it is possible to establish conditions on (37) which will assure a stable optimum system. In particular, it will be shown that sufficient conditions for the stability of the optimum system are given by the relations

$$M_{n-k} > 0 \quad (k=0,\ 1,\ \cdots,\ n-1) \tag{42}$$

where M_{n-k} are the quantities defined in equations (41).

Since the roots of (37) with positive real parts were discarded, the only possibility of obtaining an unstable system for the class of plants being considered would be the case in which equation (37) has one or more pairs of pure imaginary roots. In all other cases, the characteristic equation of the resulting system would have only roots with negative real parts. Therefore, in order to show that (42) is sufficient to insure a stable system, it is necessary to show that (42) guarantees that equation (37) has no pure imaginary roots.

First, substituting

$$\lambda^2 = Z \tag{43}$$

in equation (37) yields

$$\begin{aligned} Z^n - M_n Z^{n-1} + M_{n-1} Z^{n-2} + \cdots + M_{n-k}(-1)^{k+1} Z^{n-k-1} \\ + \cdots + (-1)^n M_1 = f(Z) = 0 \end{aligned} \tag{44}$$

Substitution of $-Z$ for Z in equation (44) gives for the general term of $f(-Z)$

$$M_{n-k}(-1)^n Z^{n-k-1} \tag{45}$$

while the first term would be

$$(-1)^n Z^n \tag{46}$$

Since all coefficients of terms of f(Z) have the same sign if all the M_{n-k} are positive, then from Descartes rule of signs, (44) can have no negative real roots. That is, if the M_{n-k} are all positive, λ^2 cannot be a negative real number, and equation (37) cannot have pure imaginary roots. But, if equation (37) has only roots with non-zero real parts, a stable optimum system is assured. The condition that all the M_{n-k} be positive is less severe than the condition given by Kalman and Bertram (1) that each c_i be greater than zero. It should be noted, however, that the condition given by Kalman and Bertram applies to a much larger class of problems than is being considered here.

The effect upon the optimum system when one or more of the M_{n-k} becomes very large is also of some interest. Consider first the case in which M_1 is very large compared to the other M_{n-k}. Then the roots can be approximated by

$$\lambda^{2n} + (-1)^n M_1 = 0 \tag{47}$$

and the roots of equation (37) will tend towards a circular pattern on the circle $R = |M_1|^{\frac{1}{2n}}$. The roots would be angularly symmetric about the real axis. That is, the roots would approach a Butterworth (22) pattern. Next consider the case in which

$$M_{n-k}(k=0,\ 1,\ \cdots,\ n-2)$$

is much greater than the other M_i. Then the approximation

$$\lambda^{2n} + (-1)^{k+1}M_{n-k}\,\lambda^{2(n-k-1)} = 0 \qquad (48)$$

can be used to estimate the general location of the roots. Equation (48) has 2(n-k-1) roots identically zero, while the remaining 2(k+1) roots will be the roots of

$$\lambda^{2(k+1)} + (-1)^{k+1}M_{n-k} = 0 \qquad (49)$$

The roots of (49) again approach a Butterworth pattern. It can be concluded, therefore, that if M_{n-k} is very large, then exactly n-k-1 roots of the optimum system characteristic equation will tend toward the origin. For example, if n = 3, then the tendency as M_3 is made larger is for one root to move away from the origin while the other two roots will tend toward the origin. However, if M_2 is made very large for n = 3, then two roots tend to become large while the third root tends toward the origin.

It is of particular interest to note that if M_3

($n \geq 3$) is made very large, then two roots of the optimum system characteristic equation will tend toward the origin while all others will tend away from the origin. If M_3 is made large enough, the roots approaching the origin will dominate the transient solution of the differential equation, and it can therefore be concluded that in order to obtain an essentially second order system it is necessary that M_3 be made large for any case in which $n \geq 3$.

These effects upon the roots of the characteristic equation are shown graphically for third order systems in Figures 2, 3, and 4, pages 38, 39, and 40. In these figures, two of the M_{n-k} have been set equal to one, and the remaining M_{n-k} has been varied from 0 to 100. It can be seen that the roots of the characteristic equation perform in the manner predicted.

In the case of a second order plant, the characteristic equation of the matrix A is given by

$$\lambda^4 - M_2 \lambda^2 + M_1 = 0 \tag{50}$$

It is possible to express the damping ratio (ζ) of the optimum closed loop system as a function of the parameters M_1 and M_2 for the second order plants. By writing the characteristic equation of a second order optimum system in the form

$$s^2+(\lambda_1+\lambda_2)s+\lambda_1\lambda_2 = s^2+2\zeta\omega_n s+\omega_n^2 = 0 \tag{50}$$

and noting that

$$\begin{aligned} M_1 &= \lambda_1^2\,\lambda_2^2 \\ M_2 &= \lambda_1^2+\lambda_2^2 \end{aligned} \tag{51}$$

for a second order plant, it is possible to show that

$$\zeta = \sqrt{\frac{1}{2} + \frac{M_2}{\sqrt{M_1}}} \tag{52}$$

The damping ratio (ζ) as a function of $M_2/\sqrt{M_1}$ is shown in Figure 5, page 41.

This analysis has shown a method of determining exact values of the c_i when the system parameters and desired location of the roots of the system characteristic equation are specified. The c_i may be determined from either equations (40) or equations (41). However, if the roots of the characteristic equation of the optimum system are specified, the use of optimization theory is unnecessary. The required feedback gains can be determined directly from equations (36). Therefore, it can be concluded that optimization of a linear constant-coefficient system using the quadratic performance measure probides little, if any, advantage over the more classical design methods. However, it is possible that the information on selecting weighting coefficients for linear constant-coefficient plants optimized over an infinite time interval might be of some value in selecting weighting constants for linear constant-coefficient plants optimized over a finite interval, for linear time-varying plants, or for non-linear plants.

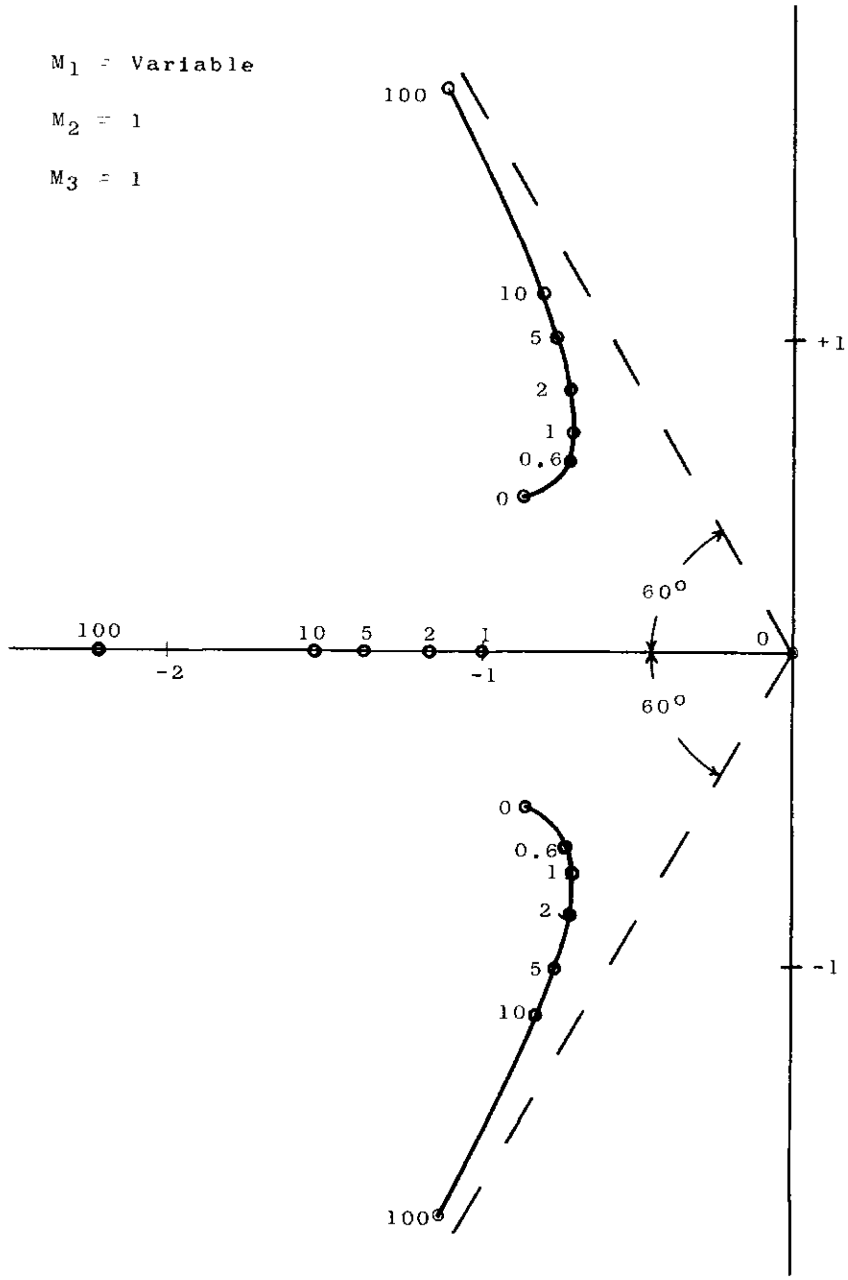

Figure 2. Root Variation as M_1 Is Varied.

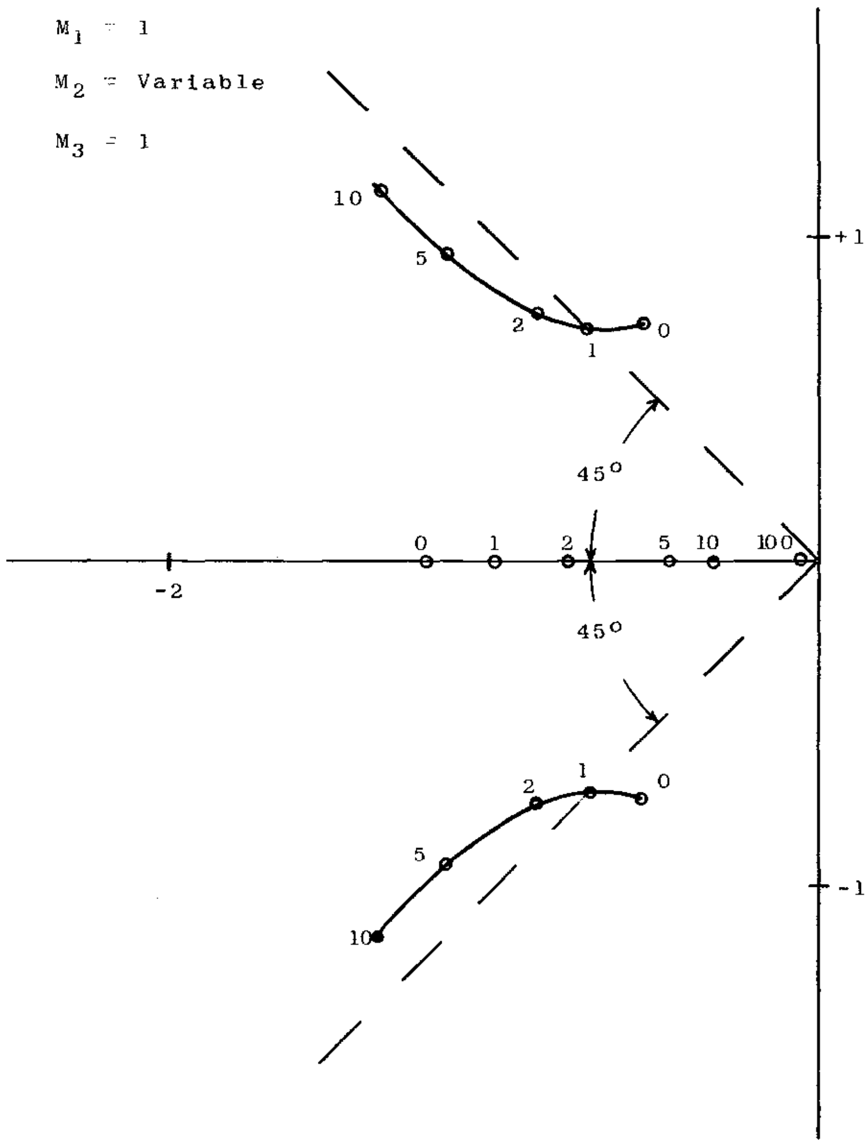

Figure 3. Root Variation as M_2 Is Varied.

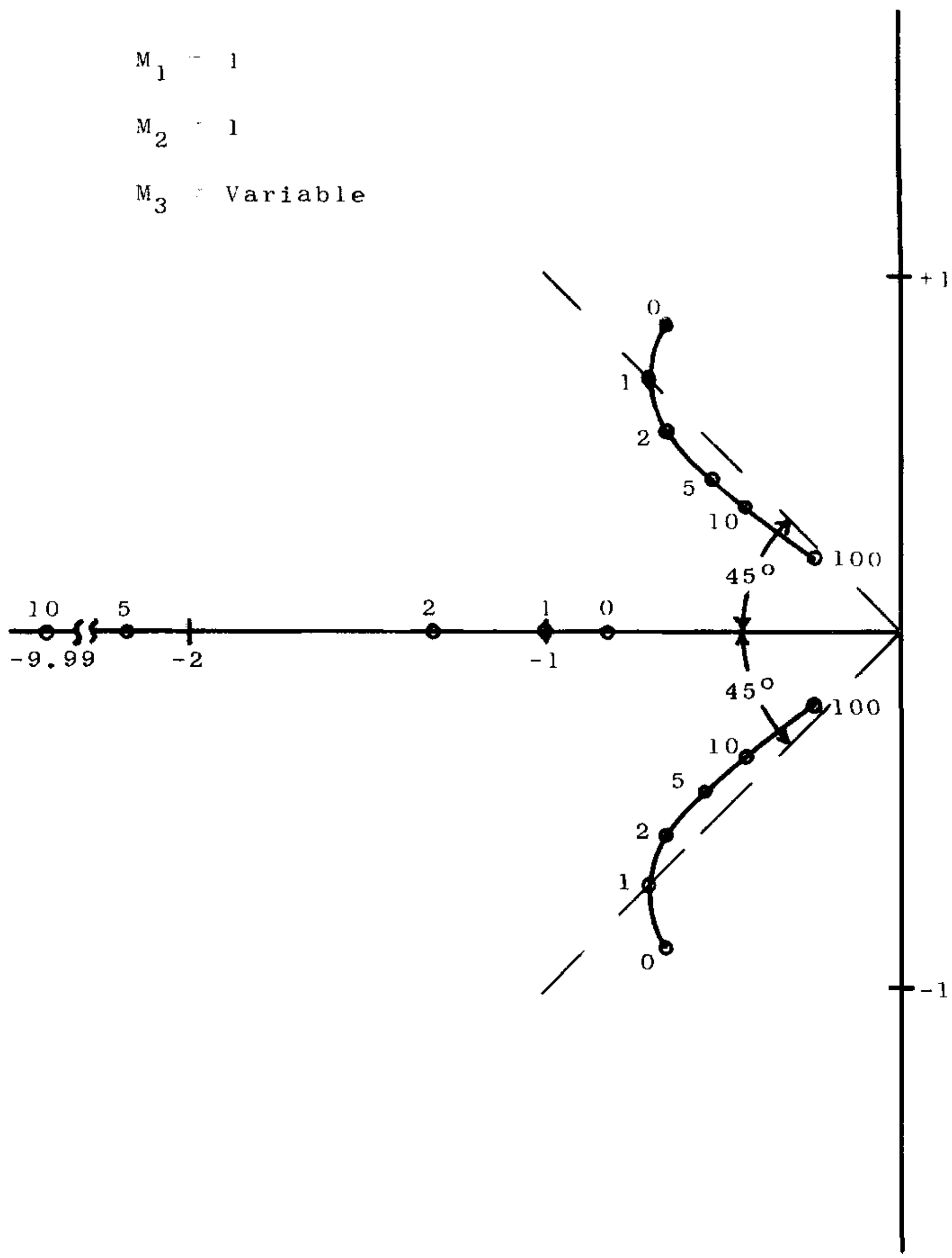

Figure 4. Root Variation as M_3 Is Varied.

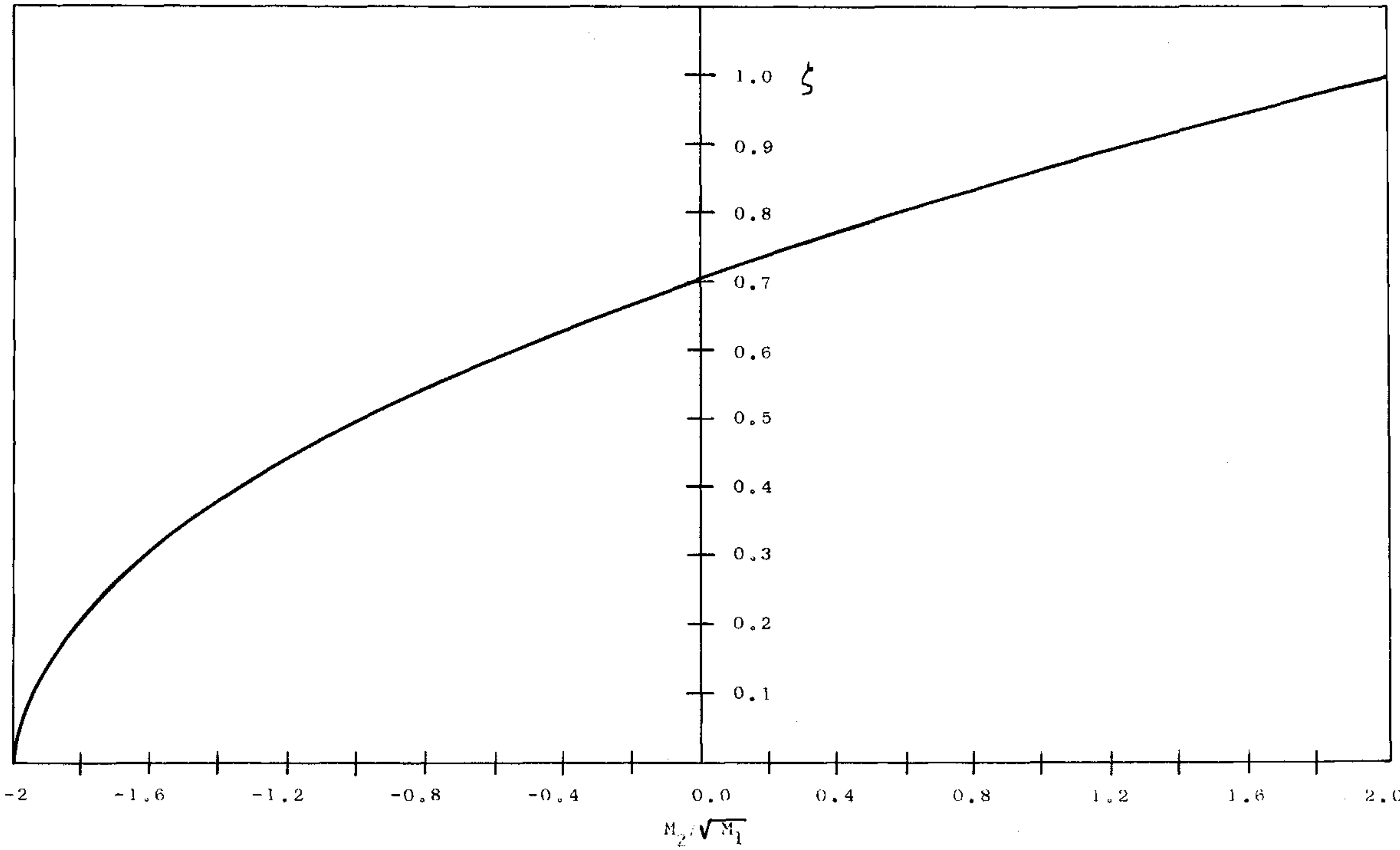

Figure 5. ζ vs. $M_2/\sqrt{M_1}$ for Second Order System.

CHAPTER III

LINEAR PLANTS OPTIMIZED OVER A FINITE TIME INTERVAL

In Chapter II the analysis was confined to linear constant coefficient plants optimized over an infinite time interval. This chapter will be devoted to linear constant coefficient plants optimized over a finite interval of time. That is, the case where T is finite in

$$IP = \frac{1}{2} \int_{0}^{T} \langle \bar{x}, C\bar{x} \rangle + \langle \bar{u}, G\bar{u} \rangle \; dt \tag{53}$$

will be examined.

The objective in this chapter will be to select the c_i for a finite time interval problem by using empirically the information on the c_i for infinite time interval problems. The transient performance of the resulting system will then be examined in order to evaluate the method of selecting the c_i, and insofar as possible to evaluate the usefulness of the quadratic IP as a design tool in this type of problem.

The determination of the optimum controller for a linear constant coefficient plant optimized over a finite time range is the same as for the infinite time range case up through equation (27) of Chapter II

$$\begin{bmatrix} \dot{\overline{x}}(t) \\ \\ \dot{\overline{p}}(t) \end{bmatrix} = \begin{bmatrix} B & D \\ \\ C & -B^T \end{bmatrix} \begin{bmatrix} \overline{x}(t) \\ \\ \overline{p}(t) \end{bmatrix} \tag{27}$$

The solution to equation (27) is shown by Coddington and Levinson (23) to be

$$\begin{bmatrix} \overline{x}(t) \\ \\ \overline{p}(t) \end{bmatrix} = \begin{bmatrix} e^{A(t-\tau)} \end{bmatrix} \begin{bmatrix} \overline{x}(\tau) \\ \\ \overline{p}(\tau) \end{bmatrix} \tag{54}$$

where $\overline{x}(\tau)$ and $\overline{p}(\tau)$ are the values of the state and costate vectors at time τ. The matrix function e^{At} is a square matrix of the same order as A, which is in this case a 2n x 2n matrix. The definition of e^{At} and some of its characteristics are given in Appendix A.

The n x n submatrices Φ_{ij} are definted by the relation

$$e^{At} = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix} \tag{55}$$

Using the above definitions in equation (54), the vector equations

$$\mathbf{x}(T) = \Phi_{11}(T-\tau)\overline{x}(\tau) + \Phi_{12}(T-\tau)\overline{p}(\tau) \qquad (56)$$

$$\overline{p}(T) = \Phi_{21}(T-\tau)\overline{x}(\tau) + \Phi_{22}(T-\tau)\overline{p}(\tau)$$

may be written. For the case in which all components of the vector $\overline{x}(t)$ are required to be zero at the final time T, the first of equations (56) becomes

$$\overline{x}(T) = [0] = \Phi_{11}(T-\tau)\overline{x}(\tau) + \Phi_{12}(T-\tau)\overline{p}(\tau) \qquad (57)$$

Solving equation (56) for $p(\tau)$ gives

$$\overline{p}(\tau) = -\Phi_{12}^{-1}(T-\tau)\Phi_{11}(T-\tau)\overline{x}(\tau) \qquad (58)$$

Taking τ as the current time, equation (58) gives the current value of the costate vector in terms of the current value of the state vector. Eliminating $\overline{p}(\tau)$ from equation (19) gives

$$\overline{u}(\tau) = -G^{-1}D^T\Phi_{12}^{-1}(T-\tau)\Phi_{11}(T-\tau)\overline{x}(\tau) \qquad (59)$$

This is the equation for the control vector $\overline{u}(\tau)$ in terms of the current value of $\overline{x}(\tau)$, which is the desired result.

The above analysis is essentially that given by Collina and Dorato in reference (18).

Example Problem

Consider a plant governed by the differential equation

$$\ddot{x} + \dot{x} = u \tag{60}$$

A control u is to be found that will bring the plant to the state

$$x(10) = \dot{x}(10) = 0 \tag{61}$$

from any initial state. The control variable u will be allowed to take on any finite value.

In order to determine u, the design will be carried out by minimizing the quadratic performance measure

$$IP = \frac{1}{2}\int_{o}^{10} (c_1x^2 + c_2\dot{x}^2 + u^2)\,dt \tag{62}$$

where c_1 and c_2 are to be chosen in some "reasonable" manner.

Relationships between M_1, M_2, ζ and ω_n for second order plants optimized over an infinite time interval were given in Chapter II. These relationships provide a method of selection the c_i for this problem. By specifying values of ζ and ω_n, values of c_1 and c_2 can be calculated for use in this case.

Since a damping ratio of 0.707 results in a linear constant coefficient system with little overshoot, a ζ of

0.707 will be used. By taking a settling time equal to the final time T, a value of ω_n may then be selected. From the usual definition of settling time

$$t_s = \frac{4}{\zeta \omega_n} \tag{63}$$

the value of ω_n is found to be

$$\omega_n = 0.567 \text{ sec.}^{-1} \tag{64}$$

From Figure 5, the value of $M_2/\sqrt{M_1}$ is found to be zero. Also, it can be seen from equations (50) and (51) that

$$M_1 = \omega_n{}^4 \tag{65}$$

The values of M_1 and M_2 are then

$$M_1 = 0.103 \tag{66}$$

$$M_2 = 0.0$$

From equation (60) the values of b_1 and b_2 are

$$b_1 = 0 \tag{67}$$

$$b_2 = 1$$

This gives for c_1 and c_2 (see Table 2)

$$c_1 = 0.103 \tag{68}$$

$$c_2 = -1.0$$

Therefore, the plant will be designed using the weighting constants

$$c_1 = 0.10 \qquad (69)$$
$$c_2 = -1.0$$

These values of c_1 and c_2 may now be used to determine $u(\tau)$ from equation (59). The solution requires a large amount of algebra, and the details are given Appendix B. The control is found to be

$$u(\tau) = -\left\{ \frac{0.16(\cosh W - \cos W)}{0.50(\cos W + \cosh W) - 1.0} x(\tau) + \frac{1.0 - 0.707(\sinh W - \sin W) + 0.50(\cos W + \cosh W)}{0.50(\cosh W + \cos W) - 1.0} \dot{x}(\tau) \right\} \qquad (70)$$

where $W = 0.8(T-\tau)$. The transient performance of the system for

$$x(0) = 1.0 \qquad (71)$$
$$\dot{x}(0) = 0.0$$

is shown in Figure 6, page 48.

The transient performance of the corresponding infinite time interval system was calculated for comparison, and it was found to differ so little from the finite time interval system that the difference could not be shown on

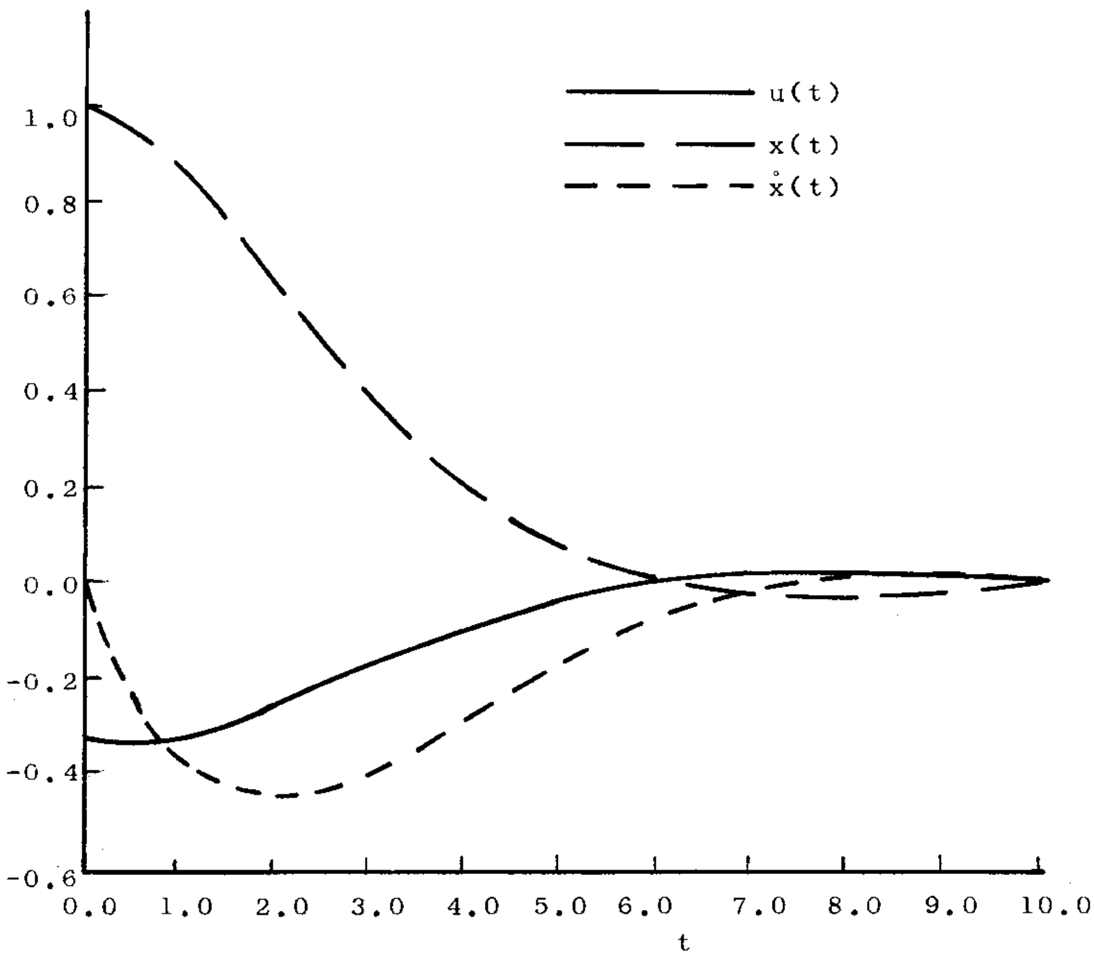

Figure 6. Response and Control Function for Optimum Finite Time Interval System.

Figure 6. Therefore, nothing is to be gained by using the optimum system designed with the values of c_i selected in this example. Simple linear feedback of displacement and rate would yield essentially the same transient response as the more complex "optimum" system.

A More General Result for The Finite Time-Interval Problem For Second Order Systems

The fact that the design obtained in the previous example did not offer any improvement in response over a constant-coefficient controller raises the question of whether it may be true in general that a linear constant coefficient controller can be found which yields essentially the same transient response as the "optimum" controller. This section is devoted to presenting at least a partial answer to the above question.

In the previous example, the values of c_1 and c_2 were determined by selecting values of ζ and ω_n for an infinite time-interval optimization of the particular plant. A value of ζ was selected which has proved successful in classical feedback control design, and the value of ω_n was then determined by specifying that

$$T = t_s = \frac{4}{\zeta\omega_n} \tag{72}$$

where T is the final time at which

$$x(T) = \dot{x}(T) = 0 \tag{73}$$

must be satisfied. Since the method used for selecting c_1 and c_2 in the previous example may have been the reason that there was no essential difference in the optimum system response and the corresponding linear system response, several arbitrary methods of selecting the c_i will be used in the following analysis. In particular, values of ζ of 0.2, 0.4 and 0.8 will be used. For each of the above values of ζ, three values of ω_n calculated from

$$T = \frac{Q}{\zeta\omega_n} \quad (Q = 1,\ 4,\ 7) \tag{74}$$

will be used. This will in effect result in nine (arbitrary) methods of determining the c_i being considered. The results will be presented in terms of the non-dimensional time $\omega_n t$.

Consider a plant governed by the differential equation

$$\frac{d^2x}{dt^2} + b_2 \frac{dx}{dt} + b_1 x = u \tag{75}$$

and the quadratic performance measure to be minimized

$$IP = \frac{1}{2}\int_0^T (c_1 x^2 + c_2 \dot{x}^2 + u^2)\, dt \tag{76}$$

subject to the conditions that

$$x(T) = \dot{x}(T) = 0 \tag{77}$$

The controller equation for this problem is found by the method of Appendix B to be

$$\begin{aligned} u(Z) = & \left\{-\omega_n^2/[(1-\zeta^2)\cosh 2\zeta(Z-z)+\zeta^2\cos 2\sqrt{1-\zeta^2}\,(Z-z)-1]\right\} \\ & \{[2\zeta^2-1+\beta_1+(1-\beta_1)(1-\zeta^2)\cosh 2\zeta(Z-z) \\ & -\zeta^2(1+\beta_1)\cos 2\sqrt{1-\zeta^2}\,(Z-z)]x(t)+[\beta_2 \\ & +\beta_2(1-\zeta^2)\cosh 2\zeta(Z-z)-\beta_2\zeta^2\cos 2\sqrt{1-\zeta^2}(Z-z) \\ & +2\zeta(1-\zeta^2)\sinh 2\zeta(Z-z)-2\zeta^2\sqrt{1-\zeta^2}\sin 2\sqrt{1-\zeta^2}(Z-z)]\dot{x}(t)\} \end{aligned} \tag{78}$$

where

$$\beta_1 = b_1/\omega_n^2 \tag{79}$$

$$\beta_2 = b_2/\omega_n \tag{80}$$

$$z = \omega_n t \tag{81}$$

$$Z = \omega_n T \tag{82}$$

Making the same substitutions in the plant differential equation results in

$$\omega_n^2\left[\frac{d^2x}{dz^2} + \beta_2\frac{dx}{dz} + \beta_1 x\right] = u \tag{83}$$

Substitution of (78) into (83) and simplifying yields the differential equation

$$\frac{d^2x}{dz^2} + \frac{2\zeta(1-\zeta^2)\sinh[2\zeta(Z-z)] - 2\zeta^2\sqrt{1-\zeta^2}\sin[2\sqrt{1-\zeta^2}(Z-z)]}{\zeta^2\cos[2\sqrt{1-\zeta^2}(Z-z)] + (1-\zeta^2)\cosh[2\zeta(Z-z)] - 1}\frac{dx}{dz} \tag{84}$$
$$+ \frac{2\zeta^2 - 1 + (1-\zeta^2)\cosh[2\zeta(Z-z)] - \zeta^2\cos[2\sqrt{1-\zeta^2}(Z-z)]}{\zeta^2\cos[2\sqrt{1-\zeta^2}(Z-z)] + (1-\zeta^2)\cosh[2\zeta(Z-z)] - 1}\, x = 0$$

which is the differential equation of the optimum closed-loop system. Note that the optimum system is described by a linear differential equation with time varying coefficients when the final time is finite. If the final time is infinite, the differential equation is linear with constant coefficients. Since

$$Z = \omega_n T = \frac{Q}{\zeta} \tag{85}$$

there are two parameters, ζ and Q, that must be selected in order to specify the differential equation. Note that selection ζ and Q is in effect the selection of values of M_1 and M_2.

The outputs $x(\omega_n t)$ of the optimum systems and of the corresponding infinite time interval system are shown in Tables 3, 4, and 5 for comparison. The tabular form of presentation was used because of the small differences between the outputs of the optimum systems with $Q = 4, 7$ and the constant coefficient systems.

Examination of the data in Tables 3, 4, and 5 indicates that for $Q = 7$ the response of the optimum system is

Table 3

Transient Responses of Certain Optimum Systems for $\zeta = 0.2$

	$X(\omega_n t)$	for	$\zeta = 0.2$	
$\omega_n t$	Q = 1	Q = 4	Q = 7	Infinite Time-Interval
0.0	1.000	1.000	1.000	1.000
1.0	0.589	0.595	0.595	0.595
2.0	-0.069	-0.127	-0.127	-0.127
3.0	-0.316	-0.515	-0.515	-0.515
4.0	-0.146	-0.384	-0.385	-0.385
5.0	0.000	-0.006	-0.006	-0.006
6.0		0.252	0.253	0.253
7.0		0.233	0.234	0.234
8.0		0.044	0.044	0.044
9.0		-0.114	-0.117	-0.117
10.0		-0.133	-0.136	-0.136
11.0		-0.045	-0.046	-0.046
12.0		0.046	0.049	0.049
13.0		0.070	0.076	0.076
14.0		0.034	0.036	0.036
15.0		-0.013	-0.018	-0.018
16.0		-0.030	-0.040	-0.040
17.0		-0.018	-0.025	-0.025
18.0		-0.002	0.004	0.004
19.0		0.002	0.021	0.021
20.0		0.000	0.016	0.016
21.0			0.001	0.001
22.0			-0.010	-0.010
23.0			-0.010	-0.010
24.0			-0.002	-0.002
25.0			0.004	0.005
26.0			0.005	0.006
27.0			0.002	0.002
28.0			-0.002	-0.002
29.0			-0.003	-0.003
30.0			-0.001	-0.002
31.0			0.000	0.001
32.0			0.001	0.002
33.0			0.001	0.001
34.0			0.000	0.000
35.0			0.000	-0.001

Table 4

Transient Responses of Certain Optimum Systems for $\zeta = 0.4$

$\omega_n t$	$X(\omega_n t)$ for $\zeta = 0.4$			
	Q = 1	Q = 4	Q = 7	Infinite Time-Interval
0.0	1.000	1.000	1.000	1.000
0.5	0.876	0.892	0.892	0.892
1.0	0.597	0.640	0.640	0.640
1.5	0.295	0.342	0.342	0.342
2.0	0.077	0.073	0.073	0.073
2.5	0.000	-0.121	-0.122	-0.122
3.0		-0.227	-0.228	-0.228
3.5		-0.252	-0.253	-0.253
4.0		-0.218	-0.219	-0.219
4.5		-0.151	-0.152	-0.152
5.0		-0.077	-0.076	-0.076
5.5		-0.012	-0.010	-0.010
6.0		0.033	0.036	0.036
6.5		0.056	0.060	0.060
7.0		0.060	0.064	0.064
7.5		0.050	0.053	0.053
8.0		0.034	0.036	0.036
8.5		0.019	0.017	0.017
9.0		0.008	0.001	0.001
9.5		0.002	-0.010	-0.010
10.0		0.000	-0.016	-0.016
10.5			-0.016	-0.016
11.0			-0.013	-0.013
11.5			-0.008	-0.008
12.0			-0.004	-0.004
12.5			0.000	0.000
13.0			0.003	0.003
13.5			0.004	0.004
14.0			0.004	0.004
14.5			0.003	0.003
15.0			0.002	0.002
15.5			0.001	0.001
16.0			0.000	0.000
16.5			0.000	-0.001
17.0			0.000	0.001
17.5			0.000	*

*Not calculated.

Table 5

Transient Responses of Certain Optimum Systems for $\zeta = 0.8$

$\omega_n t$	$x(\omega_n t)$ for $\zeta = 0.8$			
	Q = 1	Q = 4	Q = 7	Infinite Time-Interval
0.00	1.000	1.000	1.000	1.000
0.25	0.894	0.973	0.973	0.973
0.50	0.644	0.905	0.900	0.900
0.75	0.350	0.812	0.812	0.812
1.00	0.104	0.709	0.709	0.709
1.25	0.000	0.604	0.604	0.604
1.50		0.502	0.502	0.502
1.75		0.408	0.408	0.408
2.00		0.325	0.324	0.324
2.25		0.253	0.251	0.251
2.50		0.191	0.190	0.190
2.75		0.141	0.138	0.139
3.00		0.101	0.097	0.097
3.25		0.069	0.064	0.065
3.50		0.045	0.039	0.039
3.75		0.027	0.020	0.020
4.00		0.015	0.007	0.007
4.25		0.007	-0.003	-0.003
4.50		0.003	-0.009	-0.009
4.75		0.001	-0.013	-0.013
5.00		0.000	-0.015	-0.015
5.25			-0.015	-0.015
5.50			-0.015	-0.015
5.75			-0.013	-0.014
6.00			-0.012	-0.012
6.25			-0.010	-0.010
6.50			-0.009	-0.009
6.75			-0.007	-0.008
7.00			-0.005	-0.006
7.25			-0.004	-0.005
7.50			-0.003	-0.004
7.75			-0.002	-0.003
8.00			-0.001	-0.002
8.25			-0.000	-0.001
8.50			-0.000	-0.001
8.75			-0.000	*

*Not calculated.

almost identical to the response of the infinite time interval system except for the small difference in the responses near the end of the time period of interest. This is true for all values of ζ. At the final time T the optimum system will satisfy the requirement that $x(T) = 0$, while the infinite time interval system does not satisfy this requirement.

For Q = 4 the responses of the optimum and infinite time interval systems are essentially identical over the first half of the time interval of interest. During the latter half of the time inverval, the optimum system response begins to depart appreciably from that of the infinite time interval system. At the final time T, the infinite time interval system output is -0.015 for $\zeta = 0.8$ and -0.016 for $\zeta = 0.4$ and $\zeta = 0.2$.

When Q is reduced to one, the influence of the damping ratio (ζ) becomes important. The responses of the optimum and infinite time interval systems are significantly different for all values of ζ shown. This difference between the responses becomes greater as ζ is increased. The error in $x(\omega_n T)$ would probably be unacceptable for all values of ζ for which the responses are shown.

CHAPTER IV

A NON-LINEAR PLANT

The present chapter is devoted to the study of the design of a controller for a plant governed by non-linear differential equations using a quadratic IP. A desirable goal would be to determine the value of the quadratic IP as a design tool for non-linear plants. However, a more limited goal must be set for two reasons. First, it is well known that general conclusions for non-linear systems are not possible. Second, the results obtained by designing with a quadratic IP must necessarily reflect the influence of the scheme used to select the weighting coefficients. Therefore, the objective here will be to study the results of designing controllers for certain non-linear plants using the quadratic IP and a particular scheme for selecting the weighting coefficients.

The weighting coefficients in the example to follow will be selected by linearizing the plant differential equation in some "reasonable" manner, and then selecting weighting coefficients with the aid of the information obtained for linear plants in Chapter II.

Example

Consider a plant governed by the differential equation

$$\ddot{x} + 0.1\ (1-x^2)\ \dot{x} + 2x = u \tag{86}$$

A controller u is to be found that will drive the plant from

$$x(0) = 1.0 \tag{87}$$
$$\dot{x}(0) = 0.0$$

to the state

$$\left|x(3.0)\right| \leq 0.05 \tag{88}$$
$$\left|\dot{x}(3.0)\right| \leq 0.05$$

The controller is to be designed by minimizing the quadratic performance measure

$$IP = \frac{1}{2} \int_{o}^{3.0} (c_1 x^2 + c_1 \dot{x}^2 + u^2)\ dt \tag{89}$$

where c_1 and c_2 are to be selected by linearizing (86) in some manner, and choosing values of ζ and ω_n for a linearized system from consideration of the time domain performance measures that are to be satisfied.

Noting that x(0) will probably be the largest value that x(t) will attain, it appears reasonable to set x(t) equal x(0) equal 1.0 in all non-linear terms of equation

(86). The resulting linearized form of (86) is then

$$\ddot{x} + 2x = u \tag{90}$$

Assuming that a small amount of overshoot is acceptable, a damping ratio of 0.707 is specified for the corresponding linear system. Solving

$$t_s = 3.0 = \frac{4}{\zeta\omega_n} \tag{91}$$

for ω_n yields

$$\omega_n = 1.88 \tag{92}$$

In the linearized equation (86)

$$\begin{aligned} b_1 &= 2 \\ b_2 &= 0 \end{aligned} \tag{93}$$

Therefore, the solution of the equations

$$\begin{aligned} M_1 &= b_1^2 + c_1 = \omega_n^4 \\ M_2 &= b_2^2 + c_2 - 2b_1 = 0 \end{aligned} \tag{94}$$

for c_1, and c_2 results in

$$\begin{aligned} c_1 &= 8.6 \\ c_2 &= 4.0 \end{aligned} \tag{95}$$

The mathematical problem now becomes the problem of determining u(t) such that

$$IP = \frac{1}{2} \int_{o}^{3.0} (8.6x^2 + 4\dot{x}^2 + u^2)\, dt \tag{96}$$

is a minimum subject to the initial and final conditions stated in equations (87) and (88), respectively. Defining x_1 and x_2 in the usual manner

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned} \tag{97}$$

the plant differential equation may be written

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 0.1x_2 + 0.1x_1^2 x_2 + u \end{aligned} \tag{98}$$

The Hamiltonian for the problem is

$$\begin{aligned} H = {} & p_1 x_2 + p_2(-2x_1 - 0.1x_2 + 0.1x_1 x_2 + u) \\ & -\tfrac{1}{2}(8.6x_1^2 + 4x_2^2 + u^2) \end{aligned} \tag{99}$$

Taking $\frac{\partial H}{\partial u}$ and setting it equal to zero results in

$$u = p_2 \tag{100}$$

for minimum IP. The differential equations for $\dot{p}_1$ and $\dot{p}_2$ are found from

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} \tag{101}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2}$$

to be

$$\dot{p}_1 = 8.6x_1+2p_2-0.2x_1x_2p_2 \tag{102}$$

$$\dot{p}_2 = 4x_2-p_1+0.1p_2-0.1x_1^2p_2$$

Substituting p_2 for u in equation (98) yields

$$\dot{x}_1 = x_2 \tag{103}$$

$$x_2 = -2x_1-0.1x_2+0.1x_1^2x_2+p_2$$

The system of equations given by (102) and (103) must be solved subject to the conditions

$$x_1(0) = 1.0 \tag{104}$$

$$x_2(0) = 0.0$$

$$x_1(3.0) \leq 0.05$$

$$x_2(3.0) \leq 0.05$$

in order to determine the controller u that minimizes IP.

A digital computer was used to solve this system of equations. The resulting values of x(t), $\dot{x}(t)$ and u(t) for the time interval t = 0 to t = 3.0 seconds is shown in Figure 7.

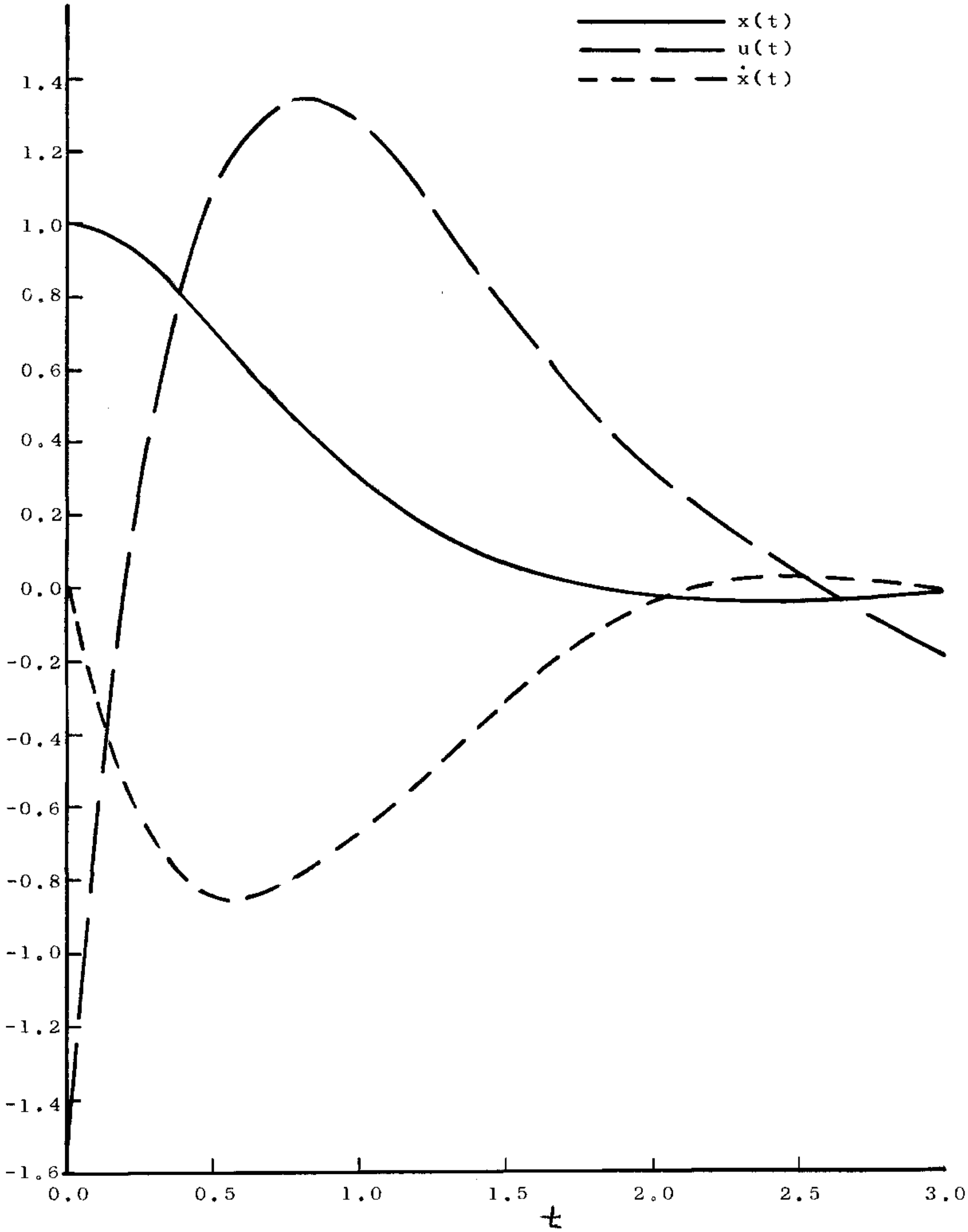

Figure 7. Optimum System Response for Non-Linear Plant.

Returning to equation (86), it can be seen that with u set to zero, the system will initially be undamped for an initial value of x equal one, while for x initially greater than one the initial damping will be negative, and for x initially less than one there would be positive damping. It is interesting to compare the control functions for these situations using the same values of c_1 and c_2 in all cases. The optimum controller and the transient response of the plant were determined for the cases

$$x(0) = 1.4 \tag{105}$$

$$\dot{x}(0) - 0.0$$

and

$$x(0) = 0.6 \tag{106}$$

$$\dot{x}(0) = 0.0$$

The same IP was used in these two cases as was used in the original case. The control functions for all three initial conditions shown in Figure 8, and the outputs x(t) are shown in Figure 9. The transient response for all three initial conditions would probably be satisfactory for many applications.

It is of some interest to compare the dynamic performance of the system above to the performance of a system

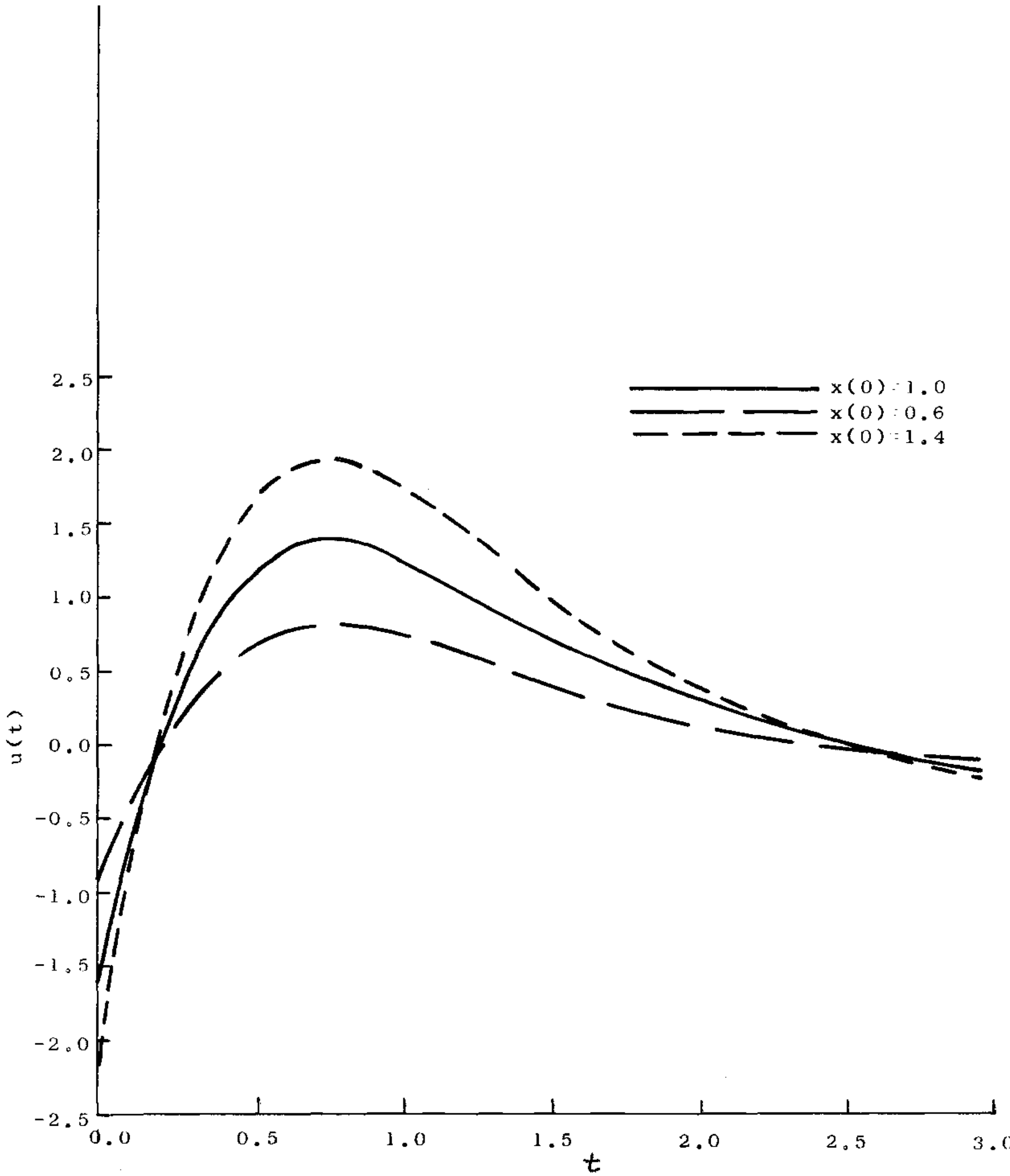

Figure 8. Control Function for Three Initial States of Non-Linear Plant.

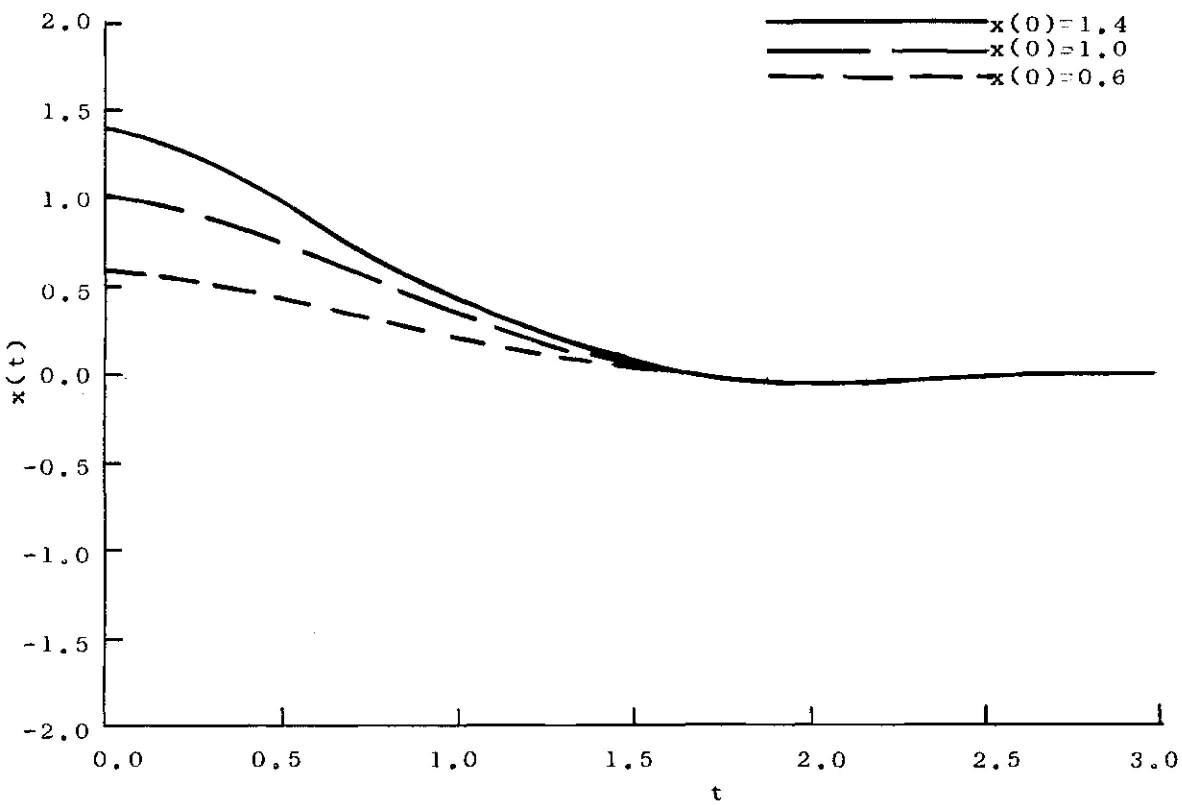

Figure 9. Transient Response for Three Initial States of Non-Linear Plant.

designed using some alternate method. A controller which results in a linear closed loop system can be found by specifying that the closed loop system satisfy the differential equation

$$\ddot{x} + 2\zeta\omega_n x + \omega_n x = 0 \qquad (107)$$

and subtracting (107) from (86) to obtain the required control function

$$u = [2-\omega_n^2]x + [0.1(1-x^2)-2\zeta\omega_n]\ \dot{x} \qquad (108)$$

The transient response and the control function for ζ and ω_n set equal to the values used in selecting c_1 and c_2 for the quadratic IP were computed for this alternate system. The transient response and the control function for this system are shown in Figure 10 along with u and x for the system designed using the quadratic IP. Examination of Figure 10 shows that the response functions of the two closed loop systems are essentially the same over the entire time interval of interest. At the final time (3.0 seconds) the value of x for the optimum system is essentially zero, while the alternate design provides a final value of $x = -0.030$ which is within the tolerences required by equation (104). The values of u for the two systems are very nearly the same for the first 2.5 seconds of the time interval of interest.

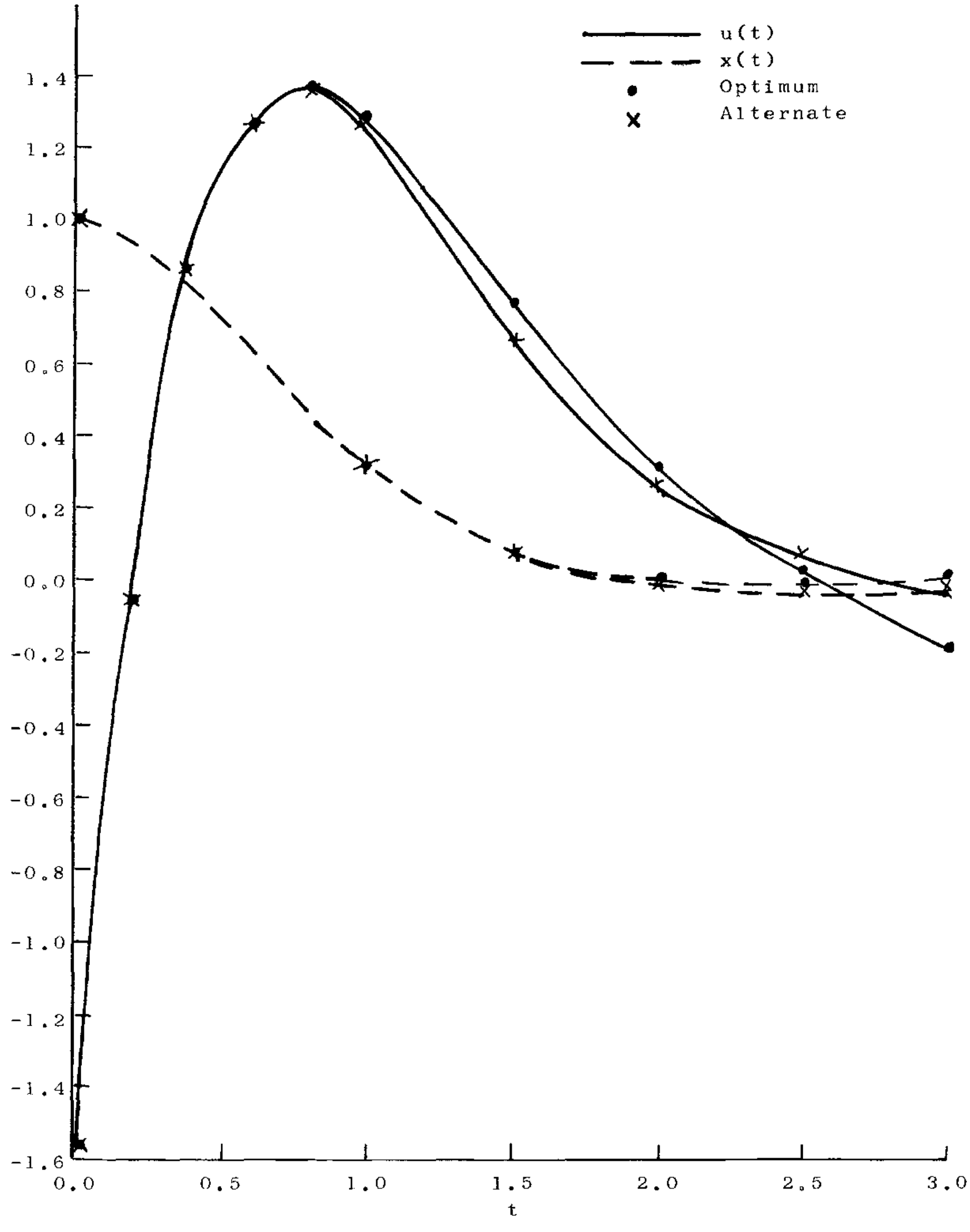

Figure 10. Comparison of Optimum Non-Linear System and Alternate System.

During the final one-half second of the time interval, the magnitude of u becomes larger for the optimum system than for the alternate system. However, this difference in the magnitudes of the control functions is not large enough to provide a basis for choosing between the two systems.

Since the transient responses and the control functions resulting from the two design methods are so very nearly the same, a choice between the two systems would have to be made upon some other basis. The cost of implementing the respective designs is one criterion that could be applied in selecting one or the other of the two designs. An on-line digital computer would be required to implement the design of the optimum system, while the alternate system would require multipliers and summers. The cost of the digital computer would in all probability be considerably greater than the cost of the multipliers and summers. Therefore, on the basis of the factors considered, the alternate system is superior to the "optimum" system.

In conclusion, while the design using the quadratic IP does produce satisfactory response, it is possible to find, in this case, a more desirable design method yielding essentially the same response at less initial cost. It should be pointed out, however, that this conclusion applies only to the example considered.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

The objective of this investigation was to study the quadratic IP and to determine insofar as possible its usefulness as a design tool in the design of automatic control systems. The problem was approached by first applying a restricted class of quadratic IP to a restricted class of linear plants. The class of IP was initially restricted to those in which the matrix of weighting coefficients was a diagonal matrix and in which the time interval was infinite. The class of plants was intially restricted to plants governed by constant coefficient linear differential equations and having a single input variable. The class of problems described above results in a final system governed by an homogeneous linear constant coefficient differential equation. For this class of problems it was possible to show relations between the elements of the weighting coefficient matrix, the plant parameters, and the roots of the characteristic equation of the optimum system. These relations were then used to select weighting coefficients for problems which were different in one way or another from those of the original class. It is possible to draw rather strong conclusions for the class of problems originally considered, but the various modifications of the original class of problems permit conclusions which are not quite so definite as those for the original class.

Since the matrix of weighting coefficients was restricted in all cases to being a diagonal matrix and the class of plants considered was restricted in all cases to those having a single input variable, it is convenient to classify the problems considered as follows:

Class I. Constant-coefficient linear plants optimized over an infinite time interval.

Class II. Constant-coefficient linear plants optimized over a finite time interval.

Class III. Non-linear plants optimized over a finite time interval.

The conclusions as to the usefulness of the quadratic IP in designing controllers may now be given for the three classes of plants considered.

For the Class I problems it was possible to show that weighting coefficients for the quadratic IP could be chosen by specifying the roots of the characteristics equation of the final system. However, if these roots are to be specified, it is senseless to deal with any IP, for the form of the controller is determined by these roots and the coefficients of the plant differential equation. Therefore, if the designer has no a priori method of selecting weighting coefficients, the use of the quadratic IP in problems of Class I is of no value whatsoever. However, if an a priori method of selecting the weighting coefficients were available, then the results of Chapter II could be used to determine the required control as a function of the state variables. The phrase "a priori method of selecting weighting coefficients" means here that the weighting coefficients may be selected without consideration of the plant differential equation or of the characteristics of the controller.

Since only second-order plants were considered from the problems included in Class II, the conclusions as to the usefulness of the quadratic IP as a design tool for this class of problems must be somewhat less definite than the conclusions for Class I problems. Comparison of the transient responses of the optimum systems indicates that, for all values of ζ, the optimum system response tends to approach the corresponding constant coefficient system response as Q is increased. As Q becomes small, the optimum system response departs significantly from the corresponding constant coefficient system response. These small values of

Q result in improved performance insofar as reduced overshoot and rise time are concerned. However, these small values of Q may require impossibly large control effort. The effect of Q upon the magnitude of control effort should be investigated further.

In terms of error at the final time T, the optimum system results in zero error while the constant coefficient system always has some error at time T. The optimum system response is therefore superior to the constant coefficient system response insofar as final error is concerned. This is true for all values of Q and ζ. However, for large Q the constant coefficient system error at time T is small compared to the initial error, and for some applications the constant coefficient system may be acceptable.

The observations above lead to the conclusion that small values of Q must be specified if the optimum system performance is to differ significantly from the performance of the corresponding constant coefficient system. However, selection of small Q values may result in excessive control effort requirements.

The conclusions that can be reached regarding problems of Class III must necessarily be less general than the conclusions that could be reached for problems of Classes I and II, because of the well known fact that it is not possible to draw general conclusions applicable to all non-linear systems. In fact, the conclusions to be drawn from the results of the study of Class III systems must be limited to the particular system studied. It should also be kept in mind during the discussion of these results that the resulting system performance depends to a large degree upon the method used in selecting the weighting coefficients and that any critique of the results of the design procedure must necessarily be a critique of the weighting coefficient selection method as well as of the IP itself.

The application of a quadratic IP to a non-linear plant resulted in a system whose transient response was reasonable, at least. Therefore, it can be concluded that the application of a quadratic IP to the problem of Class III considered, using the suggested method of selecting the weighting coefficients, results in an overall system whose transient response is satisfactory. However, it was possible to give an alternate design which produced equally satisfactory response for the example of a Class III problem. The alternate design method applied to this example resulted in a controller that would be less costly than the "optimum" controller, and therefore superior to the "optimum" controller. However, the fact that the two methods produced essentially equivalent responses and control functions could well be due to the nature of this particular problem, and this result cannot be extended to other problems of Class III at this time. The final conclusion for problems of Class III is then that the quadratic IP can result in an acceptable overall system design, but that further study is needed before any definite recommendations can be made regarding the usefulness of this IP as a design tool for this class of problems.

The conclusions reached thus far lead to the following recommendations for further study. First, an attempt should be made to find methods of specifying weighting coefficients for the quadratic IP in some a priori manner. If such methods were available, the quadratic IP would be an invaluable design tool. Second, a study of the quadratic IP applied to systems described by coupled linear constant coefficient differential equations with multiple input variables would be a natural extension of the results of Chapter II. A study of Class II problems in which the order of the plant differential equation is greater than two should also be carried out, since the conclusions drawn for Class II problems were based on results for second order systems only. A study of plants described by linear differential equations with time-varying coefficients would be desirable. This type of plant could be studied from a theoretical point of view, or some particular design problems could be solved using the quadratic IP with weighting coefficients selected in some reasonable manner. Finally, non-linear plant deserve a great deal more study because of their frequent occurrence and of the design difficulties that they present. Specifically, design of controllers for some existing non-linear plants could be carried out and evaluated.

APPENDIX A

AUXILLARY MATHEMATICAL RESULTS

Proof That $|A-\lambda I| = |A+\lambda I|$

Consider the 2n x 2n partitioned matrix

$$A = \begin{bmatrix} B & D \\ C & -B^T \end{bmatrix} \tag{A.1}$$

where the submatrices B, C, D, and $-B^T$ are n x n matrices. Let C and D be symmetric matrices. Then

$$C^T = C \tag{A.2}$$

$$D^T = D$$

where C^T denotes the transpose of C. The characteristic equation of A may be written

$$|A-\lambda I| = [0] \tag{A.3}$$

This equation may be written in the form

$$f(\lambda) = \begin{vmatrix} B - \lambda I & D \\ C & -B^T - \lambda I \end{vmatrix} = 0 \tag{A.4}$$

Replacing λ by $-\lambda$ in (A.4) gives

$$f(-\lambda) = \left|A+\lambda I\right| = \begin{vmatrix} B+\lambda I & D \\ C & -B^T+\lambda I \end{vmatrix} \tag{A.5}$$

First, interchange the first and second columns to get

$$f(-\lambda) = (-1)^n \begin{vmatrix} D & B+\lambda I \\ -B^T+\lambda I & C \end{vmatrix} \tag{A.6}$$

Interchange the rows of (A.6) to obtain

$$f(-\lambda) = (-1)^{2n} \begin{vmatrix} -B^T+\lambda I & C \\ D & B+\lambda I \end{vmatrix} \tag{A.7}$$

Next, transpose (A.7) and change the sign of the first column of the resulting determinant to get

$$f(-\lambda) = (-1)^n \begin{vmatrix} B-\lambda I & D^T \\ -C^T & B^T + \lambda I \end{vmatrix} \tag{A.8}$$

Finally, change the sign of the second row, and replace C^T and D^T by C and D, respectively. The result is

$$f(-\lambda) = \left|A+\lambda I\right| = (-1)^{2n} \begin{vmatrix} B-\lambda I & D \\ C & -B^T-\lambda I \end{vmatrix} \tag{A.9}$$

which is identical to equation (A.4). Therefore

$$|A + \lambda I| = |A - \lambda I| \tag{A.10}$$

$$f(-\lambda) = f(\lambda)$$

for any matrix A satisfying (A.1) and (A.2).

Since $f(\lambda)$ is an even polynomial, it follows that only even powers of λ occur in $f(\lambda)$. It then follows that the roots of $f(\lambda) = 0$ occur in oppositely signed pairs.

The Matrix Function e^{At}

The matrix function e^{At} is by definition

$$e^{At} = 1 + At + \frac{A^2}{2!} t^2 + \cdots \tag{A.11}$$

$$+ \frac{A^k}{k!} t^k + \cdots$$

However, it is possible to determine a closed form of e^{At} by using the Cayley-Hamilton Theorem. An algorithm for computing the closed form of e^{At} is derived by MacMillan in reference 24. The results of MacMillan's derivation are given below. Define the matrix function A_i by

$$A_i = A - \lambda_i I \quad (i = 1, 2, \cdots, r) \tag{A.12}$$

where r is the number of distinct eigenvalues, λ_i, of A. The matrix functions $R(A_i)$ are

$$R(A_i) = \prod_{j=1}^{r}{}' A_j^{m_j} \qquad (j \neq i) \tag{A.13}$$

where m_j is the multiplicity of the j^{th} eigenvalue of A. The algebraic functions $S_i(\mu)$ are defined by

$$S_i(\mu) = \prod_{i=1}^{r}{}' \frac{1}{(\mu-\lambda_j)^{m_j}} \qquad (j \neq i) \tag{A.14}$$

$$(i,\ j = 1,\ 2,\ \cdots,\ r)$$

MacMillan has shown that e^{At} is given by

$$e^{At} = \sum_{i=1}^{r} e^{\lambda_i t} R(A_i) \Big\{ \Big[I + \frac{A_1}{1!}(t+D_\mu) + \tag{A.15}$$

$$\cdots + \frac{A_i^{(m_i-1)}}{(m_i-1)!} (t+D_\mu)^{(m_i-1)} S_i(\mu) \Big\}_{\mu=\lambda_i}$$

where D_μ signifies differentiation with respect to μ.

APPENDIX B

ALGEBRA FOR FINITE TIME INTERVAL EXAMPLE

The algebraic solution of equation (55) for the case of a second order plant with all eigenvalues of A distinct is presented here. The matrix functions A_i and $R(A_i)$ and the algebraic function $S_i(\mu)$ are defined in Appendix A.

The characteristic equation of the matrix A for a second order plant is

$$\lambda^4-(b_2^2 + c_2 - 2b_1)\lambda^2 + (b_1^2 + c_1) = 0 \tag{B.1}$$

The M_i are

$$\begin{aligned} M_1 &= b_1^2 + c_1 \\ M_2 &= b_2^2 + c_2 - 2b_1 \end{aligned} \tag{B.2}$$

Since the roots of (B.1) occur in oppositely signed pairs, it is convenient to adopt the notation

$$\begin{aligned} A_j &= A - \lambda_j I \\ A_{-j} &= A + \lambda_j I \quad (j = 1, 2, \cdots, \tfrac{r}{2}) \end{aligned} \tag{B.3}$$

where r is the total number of distinct eigenvalues of A. The $R(A_i)$ may be redefined as

$$R(A_j) = A_{-j} \prod_{\ell=1}^{r/2}{}^{1} (A_{-\ell}A_\ell)^{m_\ell} \qquad (B.4)$$

$$\ell \pm j$$

The commutative properties of A with itself and with I allow $A_{-\ell}A_\ell$ to be written

$$A_{-\ell}A_\ell = (A + \lambda_\ell I)(A - \lambda_\ell I) \qquad (B.5)$$

$$= A^2 - \lambda_\ell^2 I$$

For the case when all eigenvalues are distinct (for the second order system)

$$R(A_1) = (A+\lambda_1 I)(A^2-\lambda_2^2 I) \qquad (B.6)$$

$$R(A_{-1}) = (A-\lambda_1 I)(A^2-\lambda_2^2 I)$$

$$R(A_2) = (A+\lambda_2 I)(A^2-\lambda_1^2 I)$$

$$R(A_{-2}) = (A-\lambda_2 I)(A^2-\lambda_1^2 I)$$

Examination of the relations (B.6) indicates that

$$R(A_{-1}) = R(A_1)_{\lambda_1 \to -\lambda_1} \qquad (B.7)$$

$$R(A_2) \to R(A_1)_{\lambda_1 \leftrightarrow \lambda_2}$$

$$R(A_{-2}) \to R(A_1)_{\lambda_1 \leftrightarrow -\lambda_2}$$

These relations reduce considerably the labor required in the calculation of the $R(A_i)$, because once $R(A_1)$ has been determined, the other $R(A_i)$ may be written immediately by making the interchanges indicated in equation (B.7).

It can also be shown that

$$S_{-i}(-\lambda_i) = -S_i(\lambda_i) \tag{B.8}$$

Expanding $R(A_1)$ gives

$$R(A_1) = A^3+\lambda_1 A^2-\lambda_2^2 A-\lambda_1\lambda_2^2 I \tag{B.9}$$

The matrix A for a second order plant is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b_1 & -b_2 & 0 & 1 \\ c_1 & 0 & 0 & b_1 \\ 0 & c_2 & -1 & b_2 \end{bmatrix} \tag{B.10}$$

The c_1 and c_2 terms can be eliminated by noting that the roots of equation (B.1) are related to the coefficients of (B.1) by

$$\begin{aligned} \lambda_1^2\lambda_2^2 &= b_1^2 + c_1 \\ \lambda_1^2+\lambda_2^2 &= b_2^2 + c_2-2b_1 \end{aligned} \tag{B.11}$$

Eliminating c_1 and c_2 from A gives

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b_1 & -b_2 & 0 & 1 \\ \lambda_1^2\lambda_2^2 - b_1^2 & 0 & 0 & b_1 \\ 0 & \begin{matrix}\lambda_1^2+\lambda_2^2 \\ -b_2^2+2b_1\end{matrix} & -1 & b_2 \end{bmatrix} \tag{B.12}$$

Evaluation of equation (B.9) yields for the first two rows of $R(A_1)$

$$R(A_1) = \begin{bmatrix} b_1b_2 - \lambda_1(b_1+\lambda_2^2) & \lambda_1^2+b_1-\lambda_1 b2 & -1 & \lambda_1 \\ \lambda_1\left(b_1b_2-\lambda_1(b_1+\lambda_2^2)\right) & \lambda_1(\lambda_1+b_1-\lambda_1 b_2) & -\lambda_1 & \lambda_1^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{B.13}$$

Reference to equation (51) indicates that only the first n rows of e^{At} are needed for the solution of the problem being considered.

Define the n x n submatrices $R_{k\ell}(A_i)$ by

$$R(A_i) = \begin{bmatrix} R_{11}(A_i) & R_{12}(A_i) \\ R_{21}(A_i) & R_{22}(A_i) \end{bmatrix} \tag{B.14}$$

Let $\rho_{k\ell}(A_i)$ be the elements of $R_{11}(A_i)$. Then

$$R_{11}(A_i) = \begin{bmatrix} \rho_{11}(A_i) & \rho_{12}(A_i) \\ \\ \lambda_1\rho_{11}(A_i) & \lambda_1\rho_{12}(A_i) \end{bmatrix} \tag{B.15}$$

The submatrices $\Phi_{11}(T-\tau)$ and $\Phi_{12}(T-\tau)$ become

$$\bar{\Phi}_{11}(T-\tau) = \tag{B.16}$$

$$\left[\begin{array}{l} s_1[\rho_{11}(A_1)e^{\lambda_1(T-\tau)}-\rho_{11}(A_{-1})e^{-\lambda_1(T-\tau)}] \\ +s_2[\rho_{11}(A_2)e^{\lambda_2(T-\tau)}-\rho_{11}(A_{-2})e^{-\lambda_2(T-\tau)}] \\ \\ s_1[\lambda_1\rho_{11}(A_1)e^{\lambda_1(T-\tau)}+\lambda_1\rho_{11}(A_{-1})e^{-\lambda_1(T-\tau)}] \\ +s_2[\lambda_2\rho_{11}(A_2)e^{\lambda_1(T-\tau)}+\lambda_2\rho_{11}(A_{-2})e^{-\lambda_2(T-\tau)}] \end{array}\right.$$

$$\left.\begin{array}{l} s_1[\rho_{12}(A_1)e^{\lambda_1(T-\tau)}-\rho_{12}(A_{-1})e^{-\lambda_1(T-\tau)}] \\ +s_2[\rho_{12}(A_2)e^{\lambda_2(T-\tau)}-\rho_{12}(A_{-2})e^{-\lambda_2(T-\tau)}] \\ \\ \lambda_1 s_1[\rho_{12}(A_1)e^{\lambda_1(T-\tau)}+\rho_{12}(A_{-1})e^{-\lambda_1(T-\tau)}] \\ +\lambda_2 s_2[\rho_{12}(A_2)e^{\lambda_2(T-\tau)}+\rho_{12}(A_{-2})e^{-\lambda_2(T-\tau)}] \end{array}\right]$$

$$\bar{\Phi}_{12}(T-\tau) = \tag{B.17}$$

$$\left[\begin{matrix} -s_1[e^{\lambda_1(T-\tau)}-e^{-\lambda_1(T-\tau)}] \\ -s_2[e^{\lambda_2(T-\tau)}-e^{-\lambda_2(T-\tau)}] \\ \\ -\lambda_1 s_1[e^{\lambda_1(T-\tau)}+e^{-\lambda_1(T-\tau)}] \\ -\lambda_2 s_2[e^{\lambda_2(T-\tau)}+e^{-\lambda_2(T-\tau)}] \end{matrix}\right.$$

$$\left.\begin{matrix} \lambda_1 s_1[e^{\lambda_1(T-\tau)}+e^{-\lambda_1(T-\tau)}] \\ +\lambda_2 s_2[e^{\lambda_2(T-\tau)}+e^{-\lambda_2(T-\tau)}] \\ \\ \lambda_1^2 s_1[e^{\lambda_1(T-\tau)}-e^{-\lambda_1(T-\tau)}] \\ +\lambda_2^2 s_2[e^{\lambda_2(T-\tau)}-e^{-\lambda_2(T-\tau)}] \end{matrix}\right]$$

where $s_1 = s_1(\mu)_{\mu=\lambda_1}$ (B.18)

$s_2 = s_2(\mu)_{\mu=\lambda_2}$

The costate vector $\bar{p}(\tau)$ may now be determined by evaluating equation (50). However, only $p_n(\tau)$ is needed to determine $u(\tau)$. The labor involved may be reduced accordingly. Applying Cramer's rule to equation (50) results in

$$p_2(\tau) = \frac{\Psi_1}{\Phi_{12}}x_1(\tau)-\frac{\Psi_2}{\Phi_{12}}x_2(\tau) \tag{B.19}$$

where Ψ_i is the matrix formed by replacing the n^{th} column of $\bar{\Phi}_{12}$ with the i^{th} column of $\bar{\Phi}_{11}$.

The matrices Ψ_1 and Ψ_2 are

$$\Psi_1 = \begin{bmatrix} -s_1[e^{\lambda_1(T-\tau)}-e^{-\lambda_1(T-\tau)}] & s_1[\rho_{11}(A_1)e^{\lambda_1(T-\tau)}-\rho_{11}(A_{-1})e^{-\lambda_1(T-\tau)}] \\ -s_2[e^{\lambda_2(T-\tau)}-e^{-\lambda_2(T-\tau)}] & s_2[\rho_{11}(A_2)e^{\lambda_2(T-\tau)}-\rho_{11}(A_{-2})e^{-\lambda_2(T-\tau)}] \\ -s_1\lambda_1[e^{\lambda_1(T-\tau)}+e^{-\lambda_1(T-\tau)}] & \lambda_1 s_1[\rho_{11}(A_1)e^{\lambda_1(T-\tau)}+\rho_{11}(A_{-1})e^{-\lambda_1(T-\tau)}] \\ -s_2\lambda_2[e^{\lambda_2(T-\tau)}+e^{-\lambda_2(T-\tau)}] & \lambda_2 s_2[\rho_{11}(A_2)e^{\lambda_2(T-\tau)}+\rho_{11}(A_{-2})e^{-\lambda_2(T-\tau)}] \end{bmatrix} \tag{B.20}$$

$$\Psi_1 = \begin{bmatrix} -s_1[e^{\lambda_1(T-\tau)}-e^{-\lambda_1(T-\tau)}] & s_1[\rho_{12}(A_1)e^{\lambda_1(T-\tau)}-\rho_{12}(A_{-1})e^{-\lambda_1(T-\tau)}] \\ -s_2[e^{\lambda_2(T-\tau)}-e^{-\lambda_2(T-\tau)}] & +s_2[\rho_{12}(A_2)e^{\lambda_2(T-\tau)}-\rho_{12}(A_{-2})e^{-\lambda_2(T-\tau)}] \\ -\lambda_1 s_1[e^{\lambda_1(T-\tau)}+e^{-\lambda_1(T-\tau)}] & \lambda_1 s_1[\rho_{12}(A_1)e^{\lambda_1(T-\tau)}+\rho_{12}(A_{-1})e^{-\lambda_1(T-\tau)}] \\ -\lambda_2 s_2[e^{\lambda_2(T-\tau)}+e^{-\lambda_2(T-\tau)}] & +\lambda_2 s_2[\rho_{12}(A_2)e^{\lambda_2(T-\tau)}+\rho_{12}(A_{-2})e^{-\lambda_2(T-\tau)}] \end{bmatrix} \tag{B.21}$$

By writing

$$\lambda_1 = \omega_n + i\sqrt{1-\zeta^2}\,\omega_n$$
$$\lambda_2 = \omega_n - i\sqrt{1-\zeta^2}\,\omega_n \qquad \text{(B.22)}$$

the determinants $|\Psi_1|$, $|\Psi_2|$, and $|\Phi_{12}|$ may be evaluated to yield

$$|\Psi_1| = \frac{1}{8\zeta^2\omega_n^2(1-\zeta^2)}\Big\{2\zeta^2-1+\frac{b_1}{\omega_n^2}+\Big(1-\frac{b_1}{\omega_n^2}\Big)(1-\zeta^2) \qquad \text{(B.23)}$$
$$\cosh[2\zeta\omega_n(T-\tau)] - \zeta^2\Big(1+\frac{b_1}{\omega_n^2}\Big)\cos[2\sqrt{1-\zeta^2}\,\omega_n(T-\tau)]\Big\}$$

$$|\Psi_2| = \frac{1}{8\zeta^2\omega_n^3(1-\zeta^2)}\Big\{\frac{b_2}{\omega_n} - \frac{b_2}{\omega_n}(1-\zeta^2)\cosh[2\zeta\omega_n(T-\tau)] \qquad \text{(B.24)}$$
$$-\frac{b_2}{\omega_n}\zeta^2\cos[2\sqrt{1-\zeta^2}\,\omega_n(T-\tau)]+2\zeta(1-\zeta^2)\sinh[2\zeta\omega_n(T-\tau)]$$
$$-2\zeta^2\sqrt{1-\zeta^2}\sin[2\zeta\sqrt{1-\zeta^2}\,\omega_n(T-\tau)]\Big\}$$

$$|\Phi_{12}| = \frac{1}{8\zeta^2\omega_n^4(1-\zeta^2)}\Big\{(1-\zeta^2)\cosh[2\zeta\omega_n(T-\tau)] \qquad \text{(B.25)}$$
$$+\zeta^2\cos[2\sqrt{1-\zeta^2}\,\omega_n(T-\tau)] - 1\Big\}$$

The equation of the controller may be determined by substituting equations (B.23), (B.24), and (B.25) into equation (B.19). These equations are valid for those cases in which the matrix A has only distinct eigenvalues.

LITERATURE CITED

1. R. E. Kalman and J. E. Bertram, "Control System Analysis and Design via the Second Method of Lyapunov, I, Continuous-Time System," American Society of Mechanical Engineers, Journal of Basic Engineering, Vol. 82, No. 2, June, 1960, pp. 371-393.

2. E. J. Routh, Dynamics of a System of Rigid Bodies, Advanced Part, Sixth Edition, Dover Publications, Inc., New York, 1955.

3. H. Nyquist, "Regeneration Theory," Bell System Technical Journal, January, 1932.

4. A. M. Lyapunov, "Problème Général de la Stabilité du Mouvement," Annales de la Faculté des sciences de Toulouse, second series, Vol. 9, 1907, pp. 203-474.

5. J. E. Gibson, "How to Specify the Performance of Closed Loop Systems," Control Engineering, vol. 3, pp. 122-129, September, 1956.

6. H. W. Bode, Network Analysis and Feedback Amplifier Design, D. Van Nostrand Company, Inc., Princeton, N. J., 1945.

7. A. C. Hall, The Analysis and Synthesis of Linear Servomechanisms, The Technology Press, Massachusetts Institute of Technology, Cambridge, Mass., 1943.

8. D. Graham and R. C. Lathrop, "The Synthesis of 'Optimum' Transient Response: Criteria and Standard Forms," Transactions of the American Institute of Electrical Engineers, vol. 72, pt. 2, pp. 278-288, November, 1953.

9. J. H. Westcott, "The Minimum-Moment-of-Error-Squared Criterion: A New Performance Criterion for Servos," Proceedings, Institute of Electrical Engineers, Vol. 101, pp. 471-480, October, 1954.

10. R. E. Kalman and R. W. Koepke, "Optimal Synthesis of Linear Sampling Systems Using Generalized Performance

Indexes," Transactions ASME, Vol. 80, pp. 1820-1826, November, 1958.

11. G. J. Murphy and N. T. Bold, "Optimization Based on a Square-Error Criterion with an Arbitrary Weighting Function," Institute of Radio Engineers Transactions on Automatic Controls, vol. AC-5, pp. 24-30, January, 1960.

12. Z. V. Rekasius, "A General Performance Index for Analytical Design of Control Systems," IRE Transactions on Automatic Control, Vol. AC-6 , May, 1961.

13. W. C. Schultz and V. C. Rideout, "Control System Performance Measures, Past, Present and Future," IRE Transactions on Automatic Control, February, 1961.

14. J. Wolkvitch, Ray Magdaleno, D. McRuer, D. Graham and J. McDonnell, "Performance Criteria for Linear Constant-Coefficient Systems with Deterministic Inputs," ASD-TR-61-501, February, 1962.

15. R. E. Kalman, "When Is a Linear Control System Optimal?" Preprints of Technical Papers, Fourth Joint Automatic Control Conference, June, 1963.

16. C. W. Merriam III, Optimization Theory and the Design of Feedback Control Systems, McGraw-Hill, New York, 1964.

17. L. S. Pontryagin, et al., The Mathematical Theory of Optimal Processes, Interscience Publishers, 1962.

18. G. Collina and P. Dorato, "Application of Pontryagin's Maximum Principle: Linear Control Systems," Polytechnic Institute of Brooklyn Research Report No. PIBMRI-1015-62.

19. W. Hurewicz, Lectures on Ordinary Differential Equations, The M.I.T. Press, Massachusetts Institute of Technology, Cambridge, Mass., 1958.

20. Chang, Jen-Wei, "A Problem in the Synthesis of Optimal Systems Using Maximum Principle," Automatika i Telemekhanika (English Translation) vol. 22, No. 10, October, 1961.

21. M. Marden, The Geometry of the Zeros, American Mathematical Society, New York, 1949.

22. E. A. Guillemin, Synthesis of Passive Networks, John Wiley and Sons, Inc., New York, 1957.

23. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Company, Inc., New York, 1955.

24. W. D. MacMillan, Dynamics of Rigid Bodies, Dover Publications, Inc., New York, 1960.

VITA

Charles James Bell, Jr. was born in Greenwood, Mississippi, on January 8, 1925. His parents are Charles James Bell, Sr. and Mary Garrott Bell. He attended the Greenwood, Mississippi, City Schools, graduating in 1942. After one year at Mississippi State University, he entered the U. S. Army Air Force in which he remained until December, 1945.

He returned to Mississippi State University in 1946 and received his B.S. in Mechanical Engineering there in 1949.

Upon graduation he entered his family's landscape business. He remained in this business until January, 1951, when he was called to active military duty with the 31st Infantry Division. Upon release from active duty he worked for a few months for the construction division of DuPont. In April, 1953, he returned to the family business where he remained until the business was sold in 1958. He then was employed as an instructor at Mississippi State University until entering the Graduate School of Georgia Institute of Technology in September 1960. He received his M.S. in Mechanical Engineering in June, 1963, and remained at Georgia Institute of Technology until September, 1964. He is now at Mississippi State University.